

Star Products that can not be induced by Drinfel'd Twists

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Summary of this Thesis

We give obstructions to the existence of Drinfel'd twists on connected compact 2-dimensional symplectic manifolds. In fact, only the 2-torus permits a twist star product that deforms the symplectic structure. The main observation in the line of arguments is that a twist star product forces such a symplectic manifold to be a homogeneous space. This immediately excludes the higher pretzel surfaces $T(g)$ since they can not be structured as homogeneous spaces: by a theorem due to G. D. Mostow the Euler characteristic χ of a compact homogeneous space is non-negative, while $\chi(T(g)) = 2 - 2g$. The 2-sphere \mathbb{S}^2 is a bit more involved. A. L. Onishchik classified all connected Lie groups that act transitively and effectively on \mathbb{S}^2 up to equivalence. In particular, they are semisimple Lie groups. We prove that a twist star product on \mathbb{S}^2 induces a transitive effective action of a connected Lie group on \mathbb{S}^2 that is not semisimple, to produce an obstruction also in this situation. It is the Etingof-Schiffmann subgroup of the r -matrix corresponding to the twist.

Summary of this Thesis (in German)

In dieser Masterarbeit geben wir Obstruktionen für die Existenz von Drinfel'd Twists auf zusammenhängenden, kompakten, 2-dimensionalen symplektischen Mannigfaltigkeiten. Einzig der 2-Torus besitzt ein Twist-Sternprodukt, welches die symplektische Struktur deformiert. Das Hauptargument der Beweisführung ist dabei, dass solch eine symplektische Mannigfaltigkeit mit einem Twist-Sternprodukt bereits ein homogener Raum ist. Somit können wir sofort die höheren Brezelflächen $T(g)$ ausschließen, da diese keine homogenen Räume sind: Nach einem Satz von G. D. Mostow ist die Euler-Charakteristik χ eines kompakten homogenen Raumes nicht-negativ, während $\chi(T(g)) = 2 - 2g$. Die Argumentation für die 2-Sphäre ist etwas aufwendiger. A. L. Onishchik klassifizierte bis auf Äquivalenz alle zusammenhängenden Lie-Gruppen, welche transitiv und effektiv auf \mathbb{S}^2 wirken. Diese Lie-Gruppen sind insbesondere halbeinfach. Wir beweisen, dass ein Twist-Sternprodukt auf \mathbb{S}^2 die transitive und effektive Lie-Gruppen-Wirkung einer zusammenhängenden nicht-halbeinfachen Lie-Gruppe induziert, um auch in diesem Fall eine Obstruktion zu erhalten. Es ist die Etingof-Schiffmann-Untergruppe der r -Matrix, welche dem Twist zugehört.

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Chapter 1

Introduction

Deformation Quantization

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

—Eugene Wigner, The Unreasonable Effectiveness of Mathematics in the Natural Sciences

A fundamental strategy in applied science is approximating physical systems by mathematical formalisms. The advantage of this idea is that mathematical implications can lead to predictions on the corresponding physical system. However, one has to be careful: a mathematical theory is just an approach of nature. If it is in conflict with a reproducible measurement, the mathematical framework has to be adapted. At its best by a more general theory which includes the aspects of the old theory that are conform to the physical results. One might think of *quantum mechanics* extending *classical mechanics*. Even if the laws of classical mechanics seem more nearby in daily life, quantum mechanics is said to be the best description of nature today. This statement is based on measurements on very small scales. But there is an open question: how does the classical situation appear as a limit of the quantum mechanical system? Remark that this is a pure mathematical question, since we already know that both, the classical and the quantum world, exist. There are several approaches of *quantization*, e.g. operator formalism on Hilbert spaces (consider [31, 79]) or path integral quantization (see [41, 86]). The one we are interested in is the so-called *deformation quantization*. To explain its essence we give a concrete example following [107, Section 9.1.1].

We want to discuss an approach to describe a particle of mass m influenced by a force field $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. The position of this particle at the time $t \in \mathbb{R}$ is represented by a vector $q(t) \in \mathbb{R}^3$ as well as the momentum $p(t) = m\dot{q}(t)$ of the particle. According to Newton's second law one has

$$F(q(t)) = m\ddot{q}(t), \tag{1.0.1}$$

for any time $t \in \mathbb{R}$, where $\ddot{q}(t)$ denotes the acceleration of the particle at the time t . This second order differential equation is equivalent to the two first order differential equations

$$\dot{q}(t) = \frac{1}{m}p(t) \text{ and } \dot{p}(t) = F(q(t)). \tag{1.0.2}$$

Remark that we made a change of coordinates from $q(t) \in \mathbb{R}^3$ to $(q(t), p(t)) \in \mathbb{R}^6$. If we assume F to be conservative there is a potential $V: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$F = -\nabla V, \quad (1.0.3)$$

where $\nabla V: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denotes the gradient of V . Then one can define the *Hamilton function*

$$H: \mathbb{R}^6 \ni (q, p) \mapsto \frac{p^2}{2m} + V(q) \in \mathbb{R} \quad (1.0.4)$$

of the physical system that leads to another equivalent formulation of Newton's second law: *Hamilton's equations of motion*

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)) \text{ and } \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t)). \quad (1.0.5)$$

By introducing the anti-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in M_{6 \times 6}(\mathbb{R}) \quad (1.0.6)$$

and combined coordinates $x = (q, p)$, we are able to reduce Hamilton's equations of motion to a single first order vector-valued differential equation

$$\dot{x}(t) = \Omega((\nabla H)(x(t))). \quad (1.0.7)$$

The map

$$X_H = \Omega \nabla H: \mathbb{R}^6 \rightarrow \mathbb{R}^6 \quad (1.0.8)$$

is said to be the *Hamiltonian vector field* of H . A solution $x: \mathbb{R} \rightarrow \mathbb{R}^6$ of the corresponding flow equation

$$\dot{x}(t) = X_H(x(t)) \quad (1.0.9)$$

describes the position and momentum of the particle. By defining the *Poisson bracket*

$$\{f, g\} = \langle \nabla f, \Omega \nabla g \rangle \quad (1.0.10)$$

of two smooth real-valued functions $f, g \in \mathcal{C}^\infty(\mathbb{R}^6)$ on \mathbb{R}^6 , we can state an easy condition for a function to be a *constant of motion* of the corresponding Hamilton system: the derivative of f along a solution x of Hamilton's equations of motion reads

$$\frac{d}{dt}f(x(t)) = \langle \nabla f(x(t)), \dot{x}(t) \rangle = \langle \nabla f(x(t)), \Omega((\nabla H)(x(t))) \rangle = \{f, H\}(x(t)).$$

Thus f is a constant of motion if and only if $\{f, H\}(x(t)) = 0$ for all $t \in \mathbb{R}$. It is easy to check that $\{\cdot, \cdot\}$ is a bilinear anti-symmetric map that assigns two smooth functions on \mathbb{R}^6 another smooth function on \mathbb{R}^6 . Moreover, $\{\cdot, \cdot\}$ satisfies the *Leibniz rule*

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad (1.0.11)$$

and the *Jacobi identity*

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0, \quad (1.0.12)$$

for all $f, g, h \in \mathcal{C}^\infty(\mathbb{R}^6)$. The space of smooth real-valued functions on \mathbb{R}^6 equipped with the pointwise product of functions is said to be the *algebra of classical observables*. In deformation quantization one does not change the space of variables by passing to the quantum mechanical system. Instead of this, the multiplication is changed: one deforms the associative and commutative multiplication of the classical observables by adding perturbation terms, such that the result is still associative but not commutative any more. This has to be done in a way such that the classical limit gives back the pointwise product. One also wants the new product to satisfy the *correspondence principle*: the first perturbation term of the commutator with respect to this product has to result in the Poisson bracket of the classical system. Returning to our example, we define the *Weyl-Moyal star product*

$$(f \star g)(q, p) = \sum_{m,n=0}^{\infty} \left(\frac{i\hbar}{2} \right)^{m+n} \frac{(-1)^m}{m!n!} \left(\frac{\partial^m}{\partial p^m} \frac{\partial^n}{\partial q^n} f(q, p) \right) \left(\frac{\partial^n}{\partial p^n} \frac{\partial^m}{\partial q^m} g(q, p) \right). \quad (1.0.13)$$

This associative product is a formal power series

$$f \star g = \sum_{n=0}^{\infty} (i\hbar)^n C_n(f, g), \quad (1.0.14)$$

where C_n are bidifferential operators and \hbar denotes *Planck's constant*, the formal parameter of this series. This is the first step of deformation quantization: we obtained a noncommutative algebra $(\mathcal{C}^\infty(M)[[\hbar]], \star)$ of quantum observables. The product is not commutative and we realize that

$$f \star g = fg + \frac{i\hbar}{2} \{f, g\} \mod \hbar^2. \quad (1.0.15)$$

Thus, the classical limit $\lim_{\hbar \rightarrow 0} f \star g = fg$ gives back the pointwise product. Moreover, by defining the \star -commutator $[f, g]_\star = f \star g - g \star f$ one obtains the correspondence principle

$$\lim_{\hbar \rightarrow 0} \frac{1}{i\hbar} [f, g]_\star = \{f, g\}. \quad (1.0.16)$$

This is an a posteriori motivation to choose \hbar as the formal parameter of this formal power series. The zeroth order of \star represents the classical world while quantum effects first appear in order of Planck's constant, i.e. in the first order of \star . Higher orders of \hbar are not physically measurable. In a second and last step one has to consider *states*, i.e. $\mathbb{C}[[\hbar]]$ -linear positive functionals

$$\omega: \mathcal{C}^\infty(M)[[\hbar]] \rightarrow \mathbb{C}[[\hbar]], \quad (1.0.17)$$

and *representations* of $(\mathcal{C}^\infty(M)[[\hbar]], \star)$ on Hilbert spaces which lead to the notion of \ast -algebras. One approach is the *GNS* construction (see [104]).

In a general setting, the classical observables are encoded by the algebra of smooth functions on a Poisson manifold (see Section 3.2 for a definition). Then a star product (see Definition 4.2.3) is a deformation of the pointwise product of these functions, such that the correspondence principle leads to the Poisson bracket of the manifold. Thus, if one is interested in quantization of a classical system, one option is to consider star products on Poisson manifolds. The notion of star products goes back to the work of F. Bayen et al. [9]. S. Gutt [47] and V. G. Drinfel'd [33] discovered star products as deformations of linear Poisson structures on the dual of Lie algebras. Later B. V. Fedosov proved that there is a star product on any symplectic manifold (c.f. [40]) extending the work of M. de Wilde and P. Lecomte (c.f. [29]). The contribution of Fedosov is

remarkable, since he gives a recursive formula of a star product, while constructive proofs are rare in deformation quantization. Ideas of his proof are also used beyond symplectic geometry, e.g. on Poisson manifolds (c.f. [26]) and Kähler manifolds (c.f. [19]). By a result of M. Kontsevich (c.f. [61]) there is always a star product on a Poisson manifold. A different proof by D. E. Tamarkin appeared later (c.f. [99]). A physically more relevant approach is to consider convergent star products. This can be looked up e.g. in [17,95]. Consider also [32] for a star product approach to quantum field theory. The papers [97] and [106] overview the development of deformation quantizations and provide further references.

Drinfel'd Twists

We are interested in a more specific situation, again motivated from physics: if there is a quantization of the symmetries of a classical system, the system itself has to be quantized in a compatible way. This led to the notion of *twists* in the 1980's. It was V. G. Drinfel'd who developed the corresponding theory (see e.g. [33,34]). Symmetries are encoded by formal power series of universal enveloping algebras that act on the algebra of classical observables. The twist is an element of the tensor product of the symmetries. As a result there is a way to change the Hopf algebra structure of the symmetries such that the new structure is noncocommutative. Moreover, one can deform the product of the classical observables and obtains a noncommutative product \star in terms of the twist. If \star is a star product, we received a deformation quantization of the classical system via a twist. Thus the advantage is that any module algebra of the symmetries is deformed in an appropriate way. There has been many research on twist deformation: in [35] Drinfel'd proves that twists are in one-to-one correspondence with left-invariant star products on formal power series of smooth functions on Lie groups. Concrete examples and formulas can be found in [11] and [44], while [6] gives an extension of twists to arbitrary connections. Via formality G. Halbout provides a quantization of twists on Lie bialgebras to obtain quantum twists (c.f. [48]). Furthermore, twists enable deformation quantization e.g. for actions of Kählerian Lie groups (c.f. [16]), 3-dimensional solvable Lie groups (c.f. [12]) and the Heisenberg supergroup (c.f. [13]). In quantum field theory one uses twists to deform the Poincaré group and implement the framework of noncommutative geometry (see [59]). There are even twist approaches in string theory (consider [4,5]).

Drinfel'd introduced the twist element

$$\mathcal{F} \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]] \quad (1.0.18)$$

on the formal power series of the tensor product of the universal enveloping algebra (see Definition 4.2.1). By construction it deforms the Hopf algebra structure of $\mathcal{U}(\mathfrak{g})[[\hbar]]$. In particular, it induces a deformed coproduct and a deformed antipode. Assume that the formal power series $\mathcal{C}^\infty(M)[[\hbar]]$ of smooth functions of a Poisson manifold (M, π) are a left module algebra of $\mathcal{U}(\mathfrak{g})[[\hbar]]$ (see Definition 4.1.5) via an action \triangleright . The product on $\mathcal{C}^\infty(M)[[\hbar]]$ is the pointwise multiplication \cdot of functions. Then a star product \star on M is said to be *induced by the twist \mathcal{F}* or a *twist star product* if for every $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$ one has

$$f \star g = m(\mathcal{F}^{-1} \triangleright (f \otimes g)), \quad (1.0.19)$$

where $m: \mathcal{C}^\infty(M)[[\hbar]] \otimes \mathcal{C}^\infty(M)[[\hbar]] \ni (f \otimes g) \mapsto f \cdot g \in \mathcal{C}^\infty(M)[[\hbar]]$. For a general left Hopf algebra module algebra (\mathcal{A}, \cdot) the *twist product* defined in Eq. (1.0.19) provides a deformation of a product \cdot . But one deforms in a very controlled way: the new product \star is associative

and has the same unit. Moreover, (\mathcal{A}, \star) is a left Hopf algebra module algebra of the deformed Hopf algebra. This is a very desired situation, since we do not only deform the symmetries of a system, but the system itself in a compatible way. Thus it is not surprising that Drinfel'd twists are frequently used tools in deformation quantization. For example, there are many twist approaches on the so-called q -deformed sphere (consider [25, 46, 65, 96]). Unfortunately, there are not many classifications of twist elements around (c.f. [48]) and the existence can often not be assured, besides there are not many examples. Moreover, it is not known if twist products deform the Poisson bracket of the Poisson manifold, i.e. if the correspondence principle is valid. This means that even if there are twist products it is not clear if they are physically relevant. The aim of this thesis is to show that deformation quantization via a twist is not possible in many cases.

MAIN THEOREM: *There is no twist star product on the 2-sphere and the pretzel surfaces of genus $g > 1$ deforming a symplectic structure.* ∇

Furthermore, the theorem can be applied to any symplectic foliation of a Poisson manifold. This is no classification result, but we show the constraints of twist deformation theory. Consequently, for many algebras of classical observables one has to consider a different quantization approach. This theorem will also appear in a preprint as a collaboration with Pierre Bieliavsky, Chiara Esposito and Stefan Waldmann [15].

Outlook

There are many possibilities to generalize and expand the obstructions given in this thesis. We prove in Corollary 5.2.3 that any connected compact symplectic manifold, which inherits a twist star product, has to be a homogeneous space. Thus there are no twist star products on the higher pretzel surfaces (see Theorem 6.1.1), but the contradiction applies to any connected compact symplectic homogeneous space. In particular, one can consider connected compact symplectic manifolds with negative Euler characteristic to obtain new counterexamples immediately. Also the argumentation in the proof of Theorem 6.3.4 can be adapted. There we use the classification of effective transitive actions on the 2-sphere (see Theorem 2.4.6). The existence of a twist star product on S^2 induces a transitive effective action of a connected non-semisimple Lie group on the 2-sphere. This is a contradiction to the classification that only involves semisimple Lie groups. Thus the same argumentation holds for connected compact symplectic manifolds that only admit transitive effective actions of semisimple Lie groups. The condition of being compact can be dropped if one assures the integrability of the involved Lie algebra actions. Maybe some candidates to consider are the complex projective spaces \mathbb{CP}^n and the Lagrangian Grassmannian. Furthermore, this thesis could be helpful to give a classification of Drinfel'd twists, since it shows how obstructions apply on topological and geometrical level.

Organization

This master thesis is structured as follows: Chapter 2 starts with an introduction to Lie group actions. In particular, orbits and stabilizers of Lie group actions are treated. We also recap some essential results in the theory of differential geometry, but only refer to their proofs. The study of homogeneous spaces leads naturally to transitive actions and vice versa. We prove this correspondence in full detail, since this is the most important characterization of homogeneous

spaces for our purpose. Also three possibilities on how the 2-sphere can be structured as a homogeneous space are given. It was already mentioned that compact homogeneous spaces have non-negative Euler characteristic. We give a proof assuming slightly stronger conditions by using group-invariant de Rham cohomology and Lie group representation theory. This chapter ends by summarizing the classification of connected Lie groups that act transitively and effectively on the 2-sphere. The following chapter is more algebraic. After studying some basic Lie algebra representation theory, we consider Lie bialgebras and mention Poisson-Lie groups as their global counterpart. In particular, the dual character of Lie bialgebras is pointed out. In fact, we are interested in a more specific situation: some Lie bialgebra structures are cocycles, i.e. they are obtained as the Chevalley-Eilenberg differential of an element $r \in \mathfrak{g} \wedge \mathfrak{g}$. This naturally leads to r -matrices and the classical Yang-Baxter equation. While until now most of the theory was basic, the last section of this chapter is quite specific: we define the Etingof-Schiffmann subalgebra as the Lie subalgebra, in which a r -matrix is non-degenerate. The Etingof-Schiffmann subalgebra is never semisimple, which is an essential argument. The first section of Chapter 4 is again very algebraic. We examine the notion of Drinfel'd twists in great detail. First we focus on twists on arbitrary Hopf algebras and in the second section on twists on formal power series of the universal enveloping algebra. As mentioned in the introduction, the twist induces deformation in a more categorical frame: not only the Hopf algebra but every module algebra can be deformed. The deformed algebra multiplication is said to be a twist star product in the context of formal power series and star products. As a last statement, we view a twist on formal power series as a quantization of a r -matrix. In Chapter 5 we prove one of the main results of the thesis by connecting all previous notions and results. Assuming the Poisson bivector of a connected compact symplectic manifold to be the image of a r -matrix under a Lie algebra action, we construct a transitive Lie group action on this manifold. This Lie group is the Etingof-Schiffmann subgroup corresponding to the r -matrix. Thus this special Poisson bivector is an indicator for homogeneous spaces. In particular, this situation occurs if there exists a twist star product. Thus we prove that a connected compact symplectic manifold, that can be equipped with a twist star product, is a homogeneous space and there is a r -matrix which is non-degenerate in a Lie algebra, whose Lie group acts transitively on the homogeneous space. In the last chapter we use this result to produce obstructions to twist star products. The higher pretzel surfaces are not homogeneous spaces and the sphere only admits semisimple transitive effective Lie group actions, which stands in contradiction to the action of the Etingof-Schiffmann subgroup. Remark that there are star products on all these symplectic manifolds according to the Fedosov construction. Thus there are indeed star products that can not be induced by Drinfel'd twists. The two appendices are short introductions to Hopf algebras and semisimple Lie algebras, respectively.

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Chapter 2

Transitive Actions and Homogeneous Spaces

Homogeneous spaces are one of the central concepts of this thesis. A first and very simple definition of a homogeneous space is the following: if H is a closed subgroup of a Lie group G , the set of left cosets of H in G , i.e. $\{\{gh \mid h \in H\} \mid g \in G\}$, is said to be a homogeneous space G/H . This also works for topological groups. But with this point of view we do not see the actual strength of homogeneous spaces. The original motivation to consider them is their nature to encode symmetries of spaces. To make this idea a bit more concrete one needs the notion of Lie group actions. They are smooth functions that assign to any point of a Lie group G a diffeomorphism on a smooth manifold M in a way that respects the left multiplication on G . Also, one wants the identity on G to induce the identity map on M . There are various types of Lie group actions with many nice and useful properties and we discuss the ones that are interesting for our purpose in Section 2.1. For any Lie group action there are two fundamental structures, one on the Lie group and one on the manifold. The structure on the Lie group is the set of elements which induce diffeomorphisms that reproduce a given point x on M and it is said to be the stabilizer of x . We can prove that it is always a Lie subgroup of G . On the other hand the interesting structures on the manifold are the orbits of the action. Intuitively, everyone has an idea of what an orbit should be: one might think of the way a satellite circuits a planet or electrons an atomic nucleus in the Bohr model. There is a way to phrase these ideas also in a mathematical definition: for a fixed point x on the manifold M the orbit of x is the set of points on M that are reached by applying x to every diffeomorphism that is produced by elements of G via the action. Unfortunately, not every orbit is a smooth submanifold of M , but we prove that this is true if one stays close to the identity in G or if the Lie group is compact. A very interesting situation is obtained if there is only one orbit, i.e. if all elements of x produce the same orbit which coincides with M in this case. Such actions are said to be transitive and they connect homogeneous spaces to Lie group actions. It is the task of Section 2.2 to establish this connection. First of all, it is easy to conclude that the left cosets of a closed subgroup H of the Lie group G are exactly the orbits of the right multiplication of elements of H in G . Thus one could define a homogeneous space G/H as the orbit space of this particular action. One benefit of this definition is that one is able to build the structure of a smooth manifold on the homogeneous space as a simple consequence of the “Free and Proper Action Theorem”. But there is more: we get a whole classification of homogeneous spaces by transitive actions. On one hand there is a natural transitive action of G on any homogeneous space G/H . Surprisingly, the converse statement is also true: for any transitive action of a Lie group G on a smooth

manifold M there is a diffeomorphism between M and the homogeneous space G/G_x , where G_x denotes the stabilizer of any point x of M . Thus we are able to identify M with the orbit space G/G_x , i.e. any point of M is viewed as a left coset or orbit of an element in G with respect to G_x . This also explains why one calls G the symmetries of M which was one of our initial statements. The last perception is fundamental for the proceeding of this thesis. Instead of searching for coset or orbit spaces we are looking for transitive actions, that are easy to handle. As an example, we give three different ways to structure the 2-sphere as a homogeneous space via different Lie groups that act transitively on it. The last two sections of this chapter are then more specific classifications and indicators of homogeneous spaces with topological arguments. Section 2.3 is all about to prove that connected compact homogeneous spaces have non-negative Euler characteristic. The Euler characteristic is a topological invariant of a manifold, namely the alternating series of the dimensions of its cohomology groups. It is an integer that is known for many manifolds. In particular, the connected compact pretzel surfaces $T(g)$ of genus $g \in \mathbb{N}_0$ have Euler characteristic $2(1 - g)$. It follows that $T(g)$ is not a homogeneous space if $g > 1$ and we experience the first time a very typical spirit of this thesis: we found an obstruction, i.e. there is no Lie group that acts transitively on those manifolds. In Chapter 6 we expand these obstructions to twist star products. But first we continue with Section 2.4 and classify all transitive actions that act on the 2-sphere with some additional properties. In fact, these are the three Lie group actions that we discussed in Section 2.2 before. In particular, they are semisimple Lie groups.

2.1 Lie Group Actions

We have to assume that the reader has a basic understanding of Lie theory. The symmetries of our systems are represented by Lie groups and we often investigate their infinitesimal information, i.e. their Lie algebras. For a introduction to these two fundamental objects and their connection we refer to [22, Chapter III], [36, Chapter 1], [63, Chapter 4] or [87, Chapter 4]. Most of the time we argue with a very geometric point of view, thus the proofs often inherit tools known from differential geometry. An efficient approach to this huge topic is given in [68] while we also refer to [69] and [71]. The notion of Lie group actions is also very common, but nevertheless we present the basic definitions and results in this section since these ideas are very central in this thesis. However, we proceed in a colloquial style to accelerate the process, but also to carve out the important concepts. The major source is [83, Part I Chapter 1], but the elementary theory of Lie group actions can also be found in [54, Lecture 1 and 2], [60, Chapter 2], [68, Chapter 9] and [71, Chapter 2].

In the following M always denotes a smooth manifold and G a Lie group with identity element $e \in G$. A (*left*) *action* of G on M is a smooth map

$$\Phi: G \times M \rightarrow M \quad (2.1.1)$$

that satisfies $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ and $\Phi(e, x) = x$, for all $g, h \in G$ and $x \in M$. We further denote for any $g \in G$ the induced diffeomorphism on M by

$$\Phi_g: M \ni x \mapsto \Phi_g(x) = \Phi(g, x) \in M. \quad (2.1.2)$$

This is clearly a diffeomorphism since there is the inverse map $\Phi_{g^{-1}}$ and the smoothness of both maps follows from the smoothness of Φ . Thus one can consider

$$\tilde{\Phi}: G \ni g \mapsto \Phi_g \in \text{Diff}(M), \quad (2.1.3)$$

which maps an element of the Lie group to a diffeomorphism of M . By definition $\Phi_e = \text{id}_M$ is the identity map on M , which means $e \in \ker \tilde{\Phi}$. If we have $\ker \tilde{\Phi} = \{e\}$ the action Φ is said to be *effective* or *faithful*. Actions are often demanded to be effective since one wants different group elements to induce different diffeomorphisms and this is exactly what effective actions do. If Φ is effective and $g, h \in G$ are two different elements there is an element $x \in M$ such that $\Phi_g(x) \neq \Phi_h(x)$, since otherwise $\Phi_{h^{-1}g} = \text{id}_M$ while $h^{-1}g \neq e$, which is a contradiction to the effectiveness of Φ . In particular, this situation is fulfilled if for two different elements $g, h \in G$ there is no $x \in M$ such that $\Phi_g(x) = \Phi_h(x)$. This kind of action is said to be *free*. In general $\ker \tilde{\Phi} \subseteq G$ is a Lie subgroup. This holds because $\tilde{\Phi}$ is a group homomorphism by the action property of Φ . If $\ker \tilde{\Phi}$ is a discrete subgroup we call the action Φ *locally effective*. Thus by definition any effective action is locally effective but not vice versa. Two more notions induced by actions can be obtained by considering for any $x \in M$ the smooth map

$$\Phi_x: G \ni g \mapsto \Phi_x(g) = \Phi(g, x) \in M. \quad (2.1.4)$$

We call the image

$$G \cdot x = \{\Phi_x(g) \in M \mid g \in G\} \quad (2.1.5)$$

of G under Φ_x the *orbit* of x and the preimage

$$G_x = \{g \in G \mid \Phi_x(g) = x\} \quad (2.1.6)$$

of x through Φ_x the *stabilizer* or *isotropy group* of x . Orbits are interesting objects because they build a partition of M , i.e. M can be regarded as the disjoint union of the orbits of any action Φ on M . To prove this one first recognizes that the orbits of two different points $x, y \in M$ are either equal or disjoint, since if $(G \cdot x) \cap (G \cdot y) \neq \emptyset$ there is an $z \in (G \cdot x) \cap (G \cdot y)$, i.e. there are $g, h \in G$ such that $\Phi_x(g) = z = \Phi_y(h)$. Then for any $m \in G$ the point $\Phi_y(m)$ of the orbit $G \cdot y$ is an element of $G \cdot x$ since

$$\begin{aligned} \Phi_x(mh^{-1}g) &= \Phi_{mh^{-1}g}(x) = \Phi_{mh^{-1}}(\Phi(g)(x)) = \Phi_{mh^{-1}}(\Phi(h)(y)) \\ &= \Phi_{mh^{-1}h}(y) = \Phi_y(m). \end{aligned}$$

In the same fashion $G \cdot x$ is a subset of $G \cdot y$ which implies $G \cdot x = G \cdot y$. Also any point $x \in M$ lies inside one orbit, namely in its own since $\Phi_x(e) = x$. Thus we achieved the partition of M as it was stated above. This gives the possibility to define an equivalence relation on M by declaring $x \sim y$ for $x, y \in M$ if and only if x and y lie in the same orbit. We are interested in the situation in which there is only one orbit because such actions are very much related to homogeneous spaces. We call them *transitive* actions. Equivalently, one can say that an action is transitive if for each pair $(x, y) \in M \times M$ there is an element $g \in G$ such that $\Phi(g, x) = y$. One missing piece to connect transitive actions to homogeneous spaces is the second object we defined along with orbits, the stabilizer. In the next section we see that in the case of a transitive action all stabilizers are conjugate to each other and the quotient space of the Lie group and any stabilizer is isomorphic to M . This procedure is said to structure M as a homogeneous space via a transitive action. Moreover, it turns out that the concepts of homogeneous spaces and transitive actions are equivalent.

Before we study this connection we want to prove some results that already apply to arbitrary actions. The question is what kind of substructures orbits and stabilizers are. For this we consider the map Φ_x for a $x \in M$ and prove that it has constant rank. Remember that the *rank* $\text{rank}_p f$

of a smooth map $f: M \rightarrow N$ between smooth manifolds at a point $p \in M$ is the dimension of the image of the derivative $T_p f: T_p M \rightarrow T_{f(p)} N$ of f at p , i.e. $\text{rank}_p f = \dim \text{im} T_p f$. If the rank does not depend on the point p which was chosen, the *Constant Rank Theorem* (c.f. [68, Theorem 8.8]) states that each level set of f is a closed embedded submanifold of M of codimension $\text{rank}_p f$. In this case we denote the rank of f by $\text{rank} f$ since the point we choose is not relevant. A further consequence of this theorem is that

$$T_p f^{-1}(\{q\}) = \ker T_p f \quad (2.1.7)$$

holds for all $q \in N$ (c.f. [68, Lemma 8.15]). Also recall that f is a *local diffeomorphism* if for every $p \in M$ there is an open neighbourhood $U \subseteq M$ of p such that $f|_U: U \rightarrow f(U)$ is a diffeomorphism. As a consequence of the *Inverse Function Theorem* (c.f. [68, Theorem 7.10]) this definition is equivalent to the requirement that $T_p f: T_p M \rightarrow T_{f(p)} N$ is an isomorphism for any $p \in M$.

Theorem 2.1.1 *Let $\Phi: G \times M \rightarrow M$ be an action and $x \in M$ an arbitrary point. Then the following statements hold:*

- i.) *The map Φ_x has constant rank $\text{rank}(\Phi_x) = k \in \{0, \dots, \dim M\}$.*
- ii.) *The stabilizer $G_x \subseteq G$ is a Lie subgroup of codimension k and*

$$T_e G_x = \ker T_e \Phi_x. \quad (2.1.8)$$

- iii.) *There is a neighbourhood $U \subseteq G$ of the identity such that the set $U \cdot x \subseteq M$ is a k -dimensional submanifold and*

$$T_x(U \cdot x) = T_e \Phi_x(T_e G). \quad (2.1.9)$$

- iv.) *If the orbit $G \cdot x \subseteq M$ is a submanifold, then $\dim(G \cdot x) = k$.*
- v.) *If G is compact, then $G \cdot x \subseteq M$ is a submanifold of dimension k .*

PROOF: The proof is inspired by [83, Theorem I.2.1 and Theorem I.2.3]. Fix a point $x \in M$. Because Φ_x is smooth we can differentiate it at $g \in G$ and obtain a linear map $T_g \Phi_x: T_g G \rightarrow T_{\Phi_x(g)} M$ between vector spaces and a natural number $k \in \{0, \dots, \dim M\}$ such that

$$\text{rank}_g \Phi_x = \dim \text{im}(T_g \Phi_x) = k. \quad (2.1.10)$$

We show that k is independent of the element $g \in G$. Let $h \in G$ be another element. By the action property one has

$$\Phi_h(\Phi_x(g)) = \Phi(h, \Phi(g, x)) = \Phi(hg, x) = \Phi_x(L_h(g)), \quad (2.1.11)$$

where we denoted the left multiplication with the element h on G by $L_h: G \ni g \mapsto hg \in G$. Since all the involved mappings are smooth, differentiating Eq. (2.1.11) gives by the chain rule

$$T_{\Phi_x(g)} \Phi_h T_g \Phi_x = T_{L_h(g)} \Phi_x T_g L_h = T_{hg} \Phi_x T_g L_h. \quad (2.1.12)$$

Since L_h is a diffeomorphism it is also a local diffeomorphism, i.e. we have $\text{im} T_g L_h = T_{hg} G$ and since Φ_h is a diffeomorphism we get by the same argument

$$\dim \text{im}(T_{\Phi_x(g)} \Phi_h|_{\text{im} T_g \Phi_x}) = \dim(\text{im} T_g \Phi_x) = k. \quad (2.1.13)$$

These two observations together with Eq. (2.1.12) imply that $\text{rank}_{hg}\Phi_x = \dim \text{im} T_{hg}\Phi_x = \dim \text{im} T_g\Phi_x = \text{rank}_g\Phi_x$. Thus if we set $h = g'g^{-1}$ we see that the rank of Φ_x at g coincides with the rank of Φ_x at any other point $g' \in G$. Then the rank is indeed constant and the first claim is proved. As argued before the Constant Rank Theorem implies then that the preimage $\Phi_x^{-1}(\{x\}) = G_x$ is an embedded submanifold of G of codimension k and $T_e G_x = T_e \Phi_x^{-1}(\{x\}) = \ker T_e \Phi_x$. Because the stabilizer is a closed subgroup, as the preimage of a closed subset $\{x\} \subseteq M$ under a smooth function, it is also a Lie subgroup of G (c.f. [68, Theorem 20.10]). This proves the second claim. Now since Φ_x is of constant rank, the third statement is just another consequence of the Constant Rank Theorem (c.f. [68, Theorem 7.13]). Then the next statement follows since there is a countable cover of $G \cdot x$ of k -dimensional submanifolds according to iii.). Finally, according to iii.) there is a neighbourhood $U \subseteq G$ of the identity such that $U \cdot x \subseteq M$ is a submanifold. The set $C = G \setminus (UG_x)$ fulfils $(U \cdot x) \cap (C \cdot x) = \emptyset$ and $G \cdot x = (U \cdot x) \cup (C \cdot x)$. To prove the first property assume there is an element $x' \in (U \cdot x) \cap (C \cdot x)$. Then, on one hand there is a $g \in U$ such that $\Phi_x(g) = x'$ and on the other hand there is a $h \in C$ such that $\Phi_x(h) = x'$. But this means $\Phi_g(x) = \Phi_x(g) = \Phi_x(h) = \Phi_h(x)$ which implies $\Phi_{g^{-1}h}(x) = x$, i.e. $g^{-1}h \in G_x$. Thus $h \in gG_x \subseteq UG_x$ which gives the contradiction. For the second property take $x' \in G \cdot x$ and assume that $x' \notin U \cdot x$, i.e. there is a $g \in G$ such that $x' = \Phi_x(g)$, but there is no $u \in U$ such that $x' = \Phi_x(u)$. This is exactly the condition $x' \in C \cdot x$ with $C = G \setminus (UG_x)$. Since U is open and G_x closed the set $UG_x \subseteq G$ is open. That means C is closed and for this compact as a subset of a compact set G . We already mentioned that Φ_x is continuous, so it is well known that $C \cdot x = \Phi_x(C) \subseteq M$ is still compact. We proved that the intersection of $G \cdot x$ with the open set $M \setminus C \cdot x \subseteq M$ is a submanifold, i.e. $(G \cdot x) \cap (M \setminus C \cdot x) = U \cdot x$ is a submanifold. This construction is independent of the point $x \in M$. This concludes the proof. \square

Remark that $G \cdot x$ is not always a submanifold of M . One might think of the *dense winding of the torus* (c.f. [83, page 14]).

We want to stress that there is also a notion of *right actions* which works completely analog to left actions. One calls $\Phi: G \times M \rightarrow M$ a right action of G on M if for every $x \in M$ and $g, h \in G$ one has $\Phi(gh, x) = \Phi(h, \Phi(g, x))$ and $\Phi(e, x) = x$. We adapt the same notations and results we developed for left actions. It is intuitive to denote the orbit of an element $x \in M$ in the case of a right action by $x \cdot G$. In the following sections we need both kinds of actions but left actions more frequently. For this reason, we refer to left actions just as actions.

To conclude this section we mention the correspondence of Lie group actions and Lie algebra actions. A (left) *Lie algebra action* of a Lie algebra \mathfrak{g} on M is an anti-homomorphism

$$\phi: \mathfrak{g} \rightarrow \Gamma^\infty(TM) \quad (2.1.14)$$

of Lie algebras, while a right Lie algebra action is a homomorphism. Consider a Lie group action $\Phi: G \times M \rightarrow M$ and fix a $x \in M$. One can check that

$$\phi|_x = T_e \Phi_x: \mathfrak{g} \rightarrow T_x M \quad (2.1.15)$$

determines a Lie algebra action ϕ of the Lie algebra \mathfrak{g} corresponding to G on M (see [71, Section 6.2]). Then one calls ϕ the Lie algebra action corresponding to Φ . A Lie group action Φ is said to be *locally transitive* if for any $x \in M$ the map defined in Eq. (2.1.15) is surjective. For $\xi \in \mathfrak{g}$ one calls ξ_M defined for any $x \in M$ by

$$\xi_M(x) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(t\xi)}(x) = T_e \Phi_x \xi \in T_x M \quad (2.1.16)$$

the *fundamental vector field* of ξ on M . Conversely, if the flow of every vector field $\phi(\xi) \in \Gamma^\infty(TM)$ of a Lie algebra action ϕ of \mathfrak{g} on M is complete there is a Lie group action Φ of the connected Lie group corresponding to \mathfrak{g} on M such that Eq. (2.1.15) holds. This is a famous theorem by R. Palais (c.f. [71, Theorem 6.5]). In this case the Lie algebra action ϕ is said to *integrate* to the Lie group action Φ .

2.2 Homogeneous Spaces

We start this section with the definition of a homogeneous spaces, an essential object of this thesis. After clarifying the smooth structure on homogeneous spaces we discover that there is always a natural transitive action on them and the corresponding stabilizers can be easily calculated. This can also be done by using an infinitesimal point of view. What follows is the most important classification of homogeneous spaces: they are induced by transitive actions, which help us to detect a manifold M as a homogeneous space. Of course this is also a way to find obstructions to homogeneous structures on manifolds. Relevant sources are [8, Chapter 8], [50, Chapter II], [60, Chapter 2], and [71, Chapter 2] and as in the section before we are very close to [83, Part I Chapter 1].

Let $H \subseteq G$ be a closed subgroup of the Lie group G and consider the right multiplication $R: H \times G \ni (h, g) \mapsto gh \in G$. It is known that a closed subgroup of a Lie group is indeed a Lie subgroup (c.f. [68, Theorem 20.10]), thus R is a right action of H on G . The orbits of this right action are the left cosets of H in G , i.e. $g \cdot H = \{g' \in G \mid \exists h \in H \text{ such that } g' = gh\}$. Surprisingly, there is always a smooth structure on these set of left cosets. One can prove this claim directly (c.f. [83, Theorem I.3.1]), but we want to introduce a nice tool such that the proof becomes quite trivial: the *Free and Proper Action Theorem* (c.f. [91, Corollary 6.5.1]). Remember that a (left or right) action $\Phi: G \times M \rightarrow M$ is said to be *proper* if the extended map $\bar{\Phi}: G \times M \ni (g, x) \mapsto (\Phi(g, x), x) \in M \times M$ is proper, i.e. if the preimage of all compact subsets of $M \times M$ under $\bar{\Phi}$ is again compact in $G \times M$.

Theorem 2.2.1 (Free and Proper Action Theorem) *Let $\Phi: G \times M \rightarrow M$ be a (left or right) action that is free and proper. Then there is a unique way to structure the set of orbits M/G as a smooth manifold such that the natural projection*

$$\text{pr}: M \rightarrow M/G \tag{2.2.1}$$

is a smooth surjective submersion.

Now it is easy to prove that the set G/H of orbits of R has the structure of a smooth manifold. One just has to check that R is free and proper.

Theorem 2.2.2 *Let H be a closed subgroup of G . Then there is a unique way to structure the orbit space G/H as a smooth manifold such that the map*

$$\text{pr}: G \ni g \mapsto g \cdot H \in G/H \tag{2.2.2}$$

is a smooth surjective submersion.

PROOF: We already mentioned that the right multiplication $R: H \times G \ni (h, g) \mapsto gh \in G$ is a right action of the Lie subgroup H on G and the corresponding orbits are the left cosets of H in G . The action R is free, since for any $h_1, h_2 \in H$ and $g \in G$ the equation $gh_1 = R(h_1, g) = R(h_2, g) =$

gh_2 implies $h_1 = h_2$. To prove that R is proper consider $\bar{R}: H \times G \ni (h, g) \mapsto (gh, g) \in G \times G$ and an arbitrary compact subset $K \subseteq G \times G$. If we also consider the usual right multiplication $r: G \times G \ni (g_1, g_2) \mapsto g_2g_1 \in G$ on G and $\bar{r}: G \times G \ni (g_1, g_2) \mapsto (g_2g_1, g_2) \in G \times G$ one has

$$\bar{R}^{-1}(K) = (H \times G) \cap \bar{r}^{-1}(K). \quad (2.2.3)$$

Now \bar{r} is a proper map since it is a diffeomorphism and for this $\bar{r}^{-1}(K) \subseteq G \times G$ is compact. By assumption $H \subseteq G$ is closed and so is $H \times G \subseteq G \times G$. Then Eq. (2.2.3) implies that $\bar{R}^{-1}(K)$ is compact. This concludes the proof. \square

After discovering the smooth structure on G/H we introduce the following

Definition 2.2.3 (Homogeneous Space) *A homogeneous space is the set G/H of left cosets of a closed subgroup H of a Lie group G endowed with the unique smooth structure that exists according to Theorem 2.2.2.*

Let us discuss homogeneous spaces in detail. First we want to see that there is a natural action of G on G/H and that this action is transitive. The second step will be a kind of converse statement, namely that any transitive action on a manifold makes it into a homogeneous space. To achieve it, we need some preparation.

The statement that Eq. (2.2.2) defines a surjective submersion is very beneficial and we will use this in some proofs to get smoothness of maps. To do so we refer to the

Theorem 2.2.4 (Surjective Submersion Theorem) *Let M, N and P be smooth manifolds and $\pi: M \rightarrow N$ a smooth surjective submersion. Then a map $F: N \rightarrow P$ is smooth if and only if $F \circ \pi$ is smooth.*

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \searrow F \circ \pi & \\ N & \xrightarrow{F} & P \end{array} \quad (2.2.4)$$

For a proof consider [68, Theorem 4.29].

Proposition 2.2.5 *Let H be a closed subgroup of G . There is a natural transitive action*

$$G \times G/H \ni (g, g' \cdot H) \mapsto (gg') \cdot H \in G/H \quad (2.2.5)$$

of G on G/H induced by the left multiplication on G . For $g \cdot H \in G/H$ the stabilizer is

$$G_{g \cdot H} = \text{Conj}_g(H) = gHg^{-1}. \quad (2.2.6)$$

Moreover, if we denote the Lie algebras corresponding to G and H , respectively, by \mathfrak{g} and \mathfrak{h} the tangent map at $e \in G$ of the map pr defined in Eq. (2.2.2) is an isomorphism and

$$T_e \text{pr}: \mathfrak{g}/\mathfrak{h} \rightarrow T_{\text{pr}(e)}(G/H). \quad (2.2.7)$$

PROOF: First, Theorem 2.2.2 states that G/H is a manifold. Denote the map that is defined in Eq. (2.2.5) by Φ and let $g_1, g_2 \in G$ and $g \cdot H \in G/H$ be arbitrary. Then $\Phi(e, g \cdot H) = (eg) \cdot H = g \cdot H$ and

$$\Phi(g_1, \Phi(g_2, g \cdot H)) = (g_1(g_2g)) \cdot H = ((g_1g_2)g) \cdot H = \Phi(g_1g_2, g \cdot H),$$

i.e. Φ is a set theoretic action. Since the multiplication $\mu: G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$ is smooth and pr defined in Eq. (2.2.2) is smooth according to Theorem 2.2.2, the map

$$\text{pr} \circ \mu: G \times G \ni (g_1, g_2) \mapsto (g_1 g_2) \cdot H \in G/H \quad (2.2.8)$$

is smooth and the map

$$\text{id} \times \text{pr}: G \times G \ni (g_1, g_2) \mapsto (g_1, g_2 \cdot H) \in G \times (G/H) \quad (2.2.9)$$

is a smooth surjective submersion. According to Theorem 2.2.4 the smoothness of Φ follows from the commutativity of the diagram

$$\begin{array}{ccc} G \times G & & \\ \text{id} \times \text{pr} \downarrow & \searrow \text{pr} \circ \mu & \\ G \times (G/H) & \xrightarrow{\Phi} & G/H, \end{array} \quad (2.2.10)$$

which holds since

$$(\text{pr} \circ \mu)(g_1, g_2) = (g_1 g_2) \cdot H = (\Phi \circ (\text{id} \times \text{pr}))(g_1, g_2), \quad (2.2.11)$$

for all $g_1, g_2 \in G$. Thus Φ defined in Eq. (2.2.5) is indeed an action. It is also transitive since for arbitrary $g_1 \cdot H, g_2 \cdot H \in G/H$ the element $g_2 g_1^{-1} \in G$ satisfies $\Phi(g_2 g_1^{-1}, g_1 \cdot H) = g_2 \cdot H$. If we take now an arbitrary element $g \cdot H$, the corresponding stabilizer is given by

$$G_{g \cdot H} = \{g' \in G \mid g \cdot H = \Phi(g', g \cdot H) = (g'g) \cdot H\}. \quad (2.2.12)$$

We see that if this condition is satisfied there is an element $h \in H$ such that $gh = g'g$, which is equivalent to $g' = ghg^{-1} = \text{Conj}_g(h) \in \text{Conj}_g(H)$. Thus $G_{g \cdot H} \subseteq \text{Conj}_g(H)$. Conversely, any $h \in H$ satisfies

$$\Phi(ghg^{-1}, g \cdot H) = (ghg^{-1}g) \cdot H = (gh) \cdot H = g \cdot H. \quad (2.2.13)$$

This implies $G_{g \cdot H} = \text{Conj}_g(H)$. Let us come to the infinitesimal part. Since $\text{pr}: G \rightarrow G/H$ is a submersion the map $T_e \text{pr}: T_e G \rightarrow T_{\text{pr}(e)}(G/H)$ is surjective. By assumption $T_e G = \mathfrak{g}$. We first show that $\mathfrak{h} = \ker T_e \text{pr}$. For any $\xi \in \mathfrak{h}$, one has $\exp(t\xi) \in H$ for all $t \in \mathbb{R}$ and then $\text{pr}(\exp(t\xi)) = \text{pr}(e)$ for all $t \in \mathbb{R}$. Thus by the chain rule we have

$$T_e \text{pr}(\xi) = T_e \text{pr} \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) = \left. \frac{d}{dt} \right|_{t=0} \text{pr}(\exp(t\xi)) = \left. \frac{d}{dt} \right|_{t=0} \text{pr}(e) = 0. \quad (2.2.14)$$

We showed $\mathfrak{h} \subseteq \ker T_e \text{pr}$. In fact, this is sufficient since $\text{pr}: G \rightarrow G/H$ is submersive and the dimension formula implies

$$\dim(H) = \dim(G) - \dim(G/H) = \dim(G) - (\dim(G) - \dim \ker \text{pr}) = \dim \ker \text{pr}.$$

Then $\mathfrak{h} = \ker T_e \text{pr}$ follows. By dividing by the kernel one gets the isomorphism defined in Eq. (2.2.7). \square

In the last theorem we have seen that there is a natural transitive action of G on any homogeneous space G/H . The next theorem gives the converse statement: if G acts transitively on M then M can be regarded as a homogeneous space G/H where H is the stabilizer of an arbitrary point of M .

Theorem 2.2.6 *Let $\Phi: G \times M \rightarrow M$ be a transitive action. Then for any $x \in M$ the map*

$$\beta_x: G/G_x \ni g \cdot G_x \mapsto \Phi(g, x) \in M \quad (2.2.15)$$

is a diffeomorphism, which commutes with the action of G on G/G_x . Moreover, all stabilizers are isomorphic via

$$\text{Conj}_g: G_x \rightarrow G_{\Phi_g(x)}, \quad (2.2.16)$$

where $x \in M$.

PROOF: Parts of the proof and the notation are inspired by [83, Theorem I.3.3]. Let $x \in M$ be an arbitrary point. We already mentioned that $G_x \subseteq G$ is a closed subset. Then G/G_x is a homogeneous space according to Theorem 2.2.2. If we take any $g \in G$ and $h \in G_x$ we know on the one hand that $\beta_x(h \cdot G_x) = \Phi(h, x) = x$ and on the other hand that

$$\beta_x((gh) \cdot G_x) = \Phi(gh, x) = \Phi(g, \Phi(h, x)) = \Phi(g, x) = \beta_x(g \cdot G_x), \quad (2.2.17)$$

which proves that the map β_x defined in Eq. (2.2.15) is well-defined. The surjectivity of β_x follows directly from the transitivity of Φ , i.e. for any $y \in M$ there is a $g \in G$ such that $\beta_x(g \cdot G_x) = \Phi(g, x) = y$. To prove the injectivity of β_x we take $g \cdot G_x, h \cdot G_x \in G/G_x$ and assume that $\Phi(g, x) = \beta_x(g \cdot G_x) = \beta_x(h \cdot G_x) = \Phi(h, x)$. This implies

$$\Phi(g^{-1}h, x) = \Phi(g^{-1}, \Phi(h, x)) = \Phi(g^{-1}, \Phi(g, x)) = \Phi(e, x) = x, \quad (2.2.18)$$

i.e. $g^{-1}h \in G_x$ from which $h \in g \cdot G_x$ follows. This is the injectivity $g \cdot G_x = h \cdot G_x$. The smoothness of β_x follows similarly to the proof of Proposition 2.2.5 from Theorem 2.2.4 and the commutativity of the diagram

$$\begin{array}{ccc} G & & \\ \text{pr} \downarrow & \searrow \Phi_x & \\ G/G_x & \xrightarrow{\beta_x} & M, \end{array} \quad (2.2.19)$$

since the projection is a smooth surjective submersion and $\Phi_x: G \rightarrow M$ is smooth. The next step is to show that β_x is a bijective immersion. Then it has to be a diffeomorphism since we have already proven its smoothness (c.f. [68, Corollary 7.11]). Thus consider

$$T_{\text{pr}(e)}\beta_x: T_{\text{pr}(e)}G/G_x \rightarrow T_x M. \quad (2.2.20)$$

Denote the Lie algebra corresponding to G_x by \mathfrak{g}_x (remember that G_x is a Lie group according to Theorem 2.1.1 ii.) and take $\xi \in \mathfrak{g}$. We can identify an equivalence class $[\xi] \in \mathfrak{g}/\mathfrak{g}_x$ with $T_e \text{pr}[\xi] \in T_{\text{pr}(e)}G/G_x$ according to Proposition 2.2.5. Thus if we assume $[\xi] \in \ker T_{\text{pr}(e)}\beta_x$, one has $\exp(t\xi) \in G_x$ for any $t \in \mathbb{R}$, since $\exp(0\xi) = e \in G_x$ and

$$\begin{aligned} \frac{d}{dt}\Phi_x(\exp(t\xi)) &= \frac{d}{ds}\bigg|_{s=0} \Phi_{\exp(t\xi)}(\Phi_{\exp(s\xi)}(x)) \\ &= T_x \Phi_{\exp(t\xi)} \left(\frac{d}{ds}\bigg|_{s=0} \Phi_{\exp(s\xi)}(x) \right) \end{aligned}$$

$$\begin{aligned}
&= T_x \Phi_{\exp(t\xi)} \left(\frac{d}{ds} \Big|_{s=0} \beta_x(\exp(s\xi) \cdot G_x) \right) \\
&= T_x \Phi_{\exp(t\xi)} (T_{\text{pr}(e)} \beta_x[\xi]) \\
&= 0.
\end{aligned}$$

Thus $\xi \in \mathfrak{g}_x$ for which $[\xi] = 0$ and the map defined in (2.2.20) is injective. For $g \in G$, the derivative of the equality

$$\beta_x(g \cdot G_x) = \Phi(g, x) = \Phi(g, \Phi(e, x)) = \Phi_g \circ \beta_x(e \cdot G_x) = \Phi_g \circ \beta_x \circ \ell_{g^{-1}}(g \cdot G_x) \quad (2.2.21)$$

gives

$$T_{\text{pr}(g)} \beta_x = T_x \Phi_g \circ T_{\text{pr}(e)} \beta_x \circ T_g \ell_{g^{-1}}, \quad (2.2.22)$$

according to the chain rule, where $\ell_{g^{-1}}: G/G_x \ni g' \cdot G_x \mapsto (g^{-1}g') \cdot G_x \in G/G_x$. Since Φ_g and $\ell_{g^{-1}}$ are diffeomorphisms this implies the injectivity of $T_{\text{pr}(g)} \beta_x$. As we argued this is enough to show that β_x is a diffeomorphism since the involved spaces are second countable. Moreover, β_x is equivariant, i.e. it commutes with Φ , since for any $g \in G$ and $h \cdot G_x \in G/G_x$ one has

$$\Phi_g(\beta_x(h \cdot G_x)) = \Phi_g(\Phi_h(x)) = \Phi_{gh}(x) = \beta_x((gh) \cdot G_x) = \beta_x(\tilde{\Phi}_g(h \cdot G_x)),$$

where $\tilde{\Phi}$ denotes the action of G on G/G_x defined by Eq. (2.2.5). Thus $\Phi_g \circ \beta_x = \beta_x \circ \tilde{\Phi}_g$. To prove that all stabilizers are isomorphic via the map defined in Eq. (2.2.16) we choose two points $x, y \in M$. Since Φ is transitive there is a $g \in G$ such that $\Phi_x(g) = y$. If we take $g' \in G_x$, one has

$$\begin{aligned}
\Phi_{\Phi_g(x)}(\text{Conj}_g(g')) &= \Phi_{\Phi_g(x)}(gg'g^{-1}) = \Phi(gg'g^{-1}, \Phi_g(x)) = \Phi(gg', x) \\
&= \Phi(g, \Phi(g', x)) = \Phi(g, x) = \Phi_g(x),
\end{aligned}$$

which means that $\text{Conj}_g(g') \in G_{\Phi_g(x)}$. Thus the map defined in Eq. (2.2.16) is well-defined. Conj_g is obviously smooth and also invertible with inverse $\text{Conj}_{g^{-1}}: G_{\Phi_g(x)} \rightarrow G_x$. Indeed, if $g' \in G_{\Phi_g(x)}$ one gets

$$\begin{aligned}
\Phi(\text{Conj}_{g^{-1}}(g'), x) &= \Phi(g^{-1}g'g, x) = \Phi(g^{-1}, \Phi(g'g, x)) = \Phi(g^{-1}, \Phi(g', \Phi(g, x))) \\
&= \Phi(g^{-1}, \Phi_g(x)) = \Phi(e, x) = x.
\end{aligned}$$

Then $\text{Conj}_{g^{-1}}$ is a well-defined mapping from $G_{\Phi_g(x)}$ to G_x . Of course $\text{Conj}_g \circ \text{Conj}_{g^{-1}} = \text{id}_{G_{\Phi_g(x)}}$ and $\text{Conj}_{g^{-1}} \circ \text{Conj}_g = \text{id}_{G_x}$. Thus (2.2.16) is indeed an isomorphism and $G_{\Phi_g(x)} = G_y$. \square

We want to illustrate this construction by considering three examples of transitive Lie group actions on the 2-sphere

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}. \quad (2.2.23)$$

Example 2.2.7 The **special orthogonal group** in three dimensions (also called the **rotation group**) $\text{SO}(3) = \{A \in M_{3 \times 3}(\mathbb{R}) \mid AA^T = \mathbb{1} = A^T A \text{ and } \det(A) = 1\}$ is a connected compact 3-dimensional Lie group. Consider the map

$$\Phi: \text{SO}(3) \times \mathbb{S}^2 \ni (A, p) \mapsto \Phi(A, p) = Ap \in \mathbb{S}^2, \quad (2.2.24)$$

where the right side of Eq. (2.2.24) denotes the matrix-vector multiplication. It is well-defined, since

$$\|Ap\|^2 = \langle Ap, Ap \rangle = \langle A^T Ap, p \rangle = \langle p, p \rangle = \|p\|^2 = 1,$$

for any $A \in \text{SO}(3)$ and $p \in \mathbb{S}^2$. The smoothness of the linear matrix-vector multiplication induces the smoothness of Φ . Moreover, Φ is a Lie group action of $\text{SO}(3)$ on \mathbb{S}^2 since for any $A, B \in \text{SO}(3)$ and $p \in \mathbb{S}^2$ one has

$$\Phi(A, \Phi(B, p)) = A(Bp) = (AB)p = \Phi(AB, p) \text{ and } \Phi(\mathbb{1}, p) = \mathbb{1}p = p,$$

where $\mathbb{1} \in \text{SO}(3)$ denotes the identity matrix. This action is even transitive: choose two points $p \neq q$ on \mathbb{S}^2 . Then there is a unique plane E in \mathbb{R}^3 such that $0, p, q \in E$, where $0 = (0 \ 0 \ 0)^T \in \mathbb{R}^3$. Choose a vector $0 \neq n \in \mathbb{R}^3$ perpendicular to E . There is an angle $\alpha \in [0, 2\pi]$ such that a rotation of E around n transfers p onto q . After a change of coordinates we can assume that this is a rotation around the z -axis and p lies on the x -axis. Then

$$A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SO}(3) \quad (2.2.25)$$

satisfies $Ap = q$. If $p = q \in \mathbb{S}^2$ one simply has $\mathbb{1}p = q$. Altogether Φ is transitive. Since any $\mathbb{1} \neq A \in \text{SO}(3)$ has an eigenvalue $1 \neq \lambda \in \mathbb{C}$, only $\mathbb{1}$ induces the identity diffeomorphism on \mathbb{S}^2 and Φ is effective in addition. We want to compute the stabilizer at the point $e_1 = (1 \ 0 \ 0)^T \in \mathbb{S}^2$.

Thus we are searching for the elements $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \text{SO}(3)$ that satisfy $Ae_1 = e_1$. The matrix A preserves e_1 if and only if $a = 1$ and $d = g = 0$. Then $A^T A = \mathbb{1}$ implies $b = c = 0$. The conditions remaining for e, h, f, i are those for $B = \begin{pmatrix} e & f \\ h & i \end{pmatrix} \in M_{2 \times 2}(\mathbb{R})$ to be an element of $\text{SO}(2)$. This shows

$$\text{SO}(3)_{e_1} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & \text{SO}(2) \end{pmatrix}, \quad (2.2.26)$$

where $\vec{0} = (0 \ 0)^T \in \mathbb{R}^2$. Thus we can identify $\text{SO}(3)_{e_1}$ with $\text{SO}(2)$. According to Theorem 2.2.6 the 2-sphere \mathbb{S}^2 is a homogeneous space and there is a diffeomorphism

$$\text{SO}(3)/\text{SO}(2) \cong \mathbb{S}^2. \quad (2.2.27)$$

More explicit, the diffeomorphism reads

$$\beta_{e_1} : \text{SO}(3)/\text{SO}(2) \ni A \cdot \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & \text{SO}(2) \end{pmatrix} \mapsto Ae_1 \in \mathbb{S}^2. \quad (2.2.28)$$

A non-compact Lie group containing the compact Lie group $\text{SO}(3)$ is the special linear group

$$\text{SL}(3, \mathbb{R}) = \{A \in M_{3 \times 3}(\mathbb{R}) \mid \det(A) = 1\}. \quad (2.2.29)$$

One can extend the linear action of $\text{SO}(3)$ on \mathbb{S}^2 to $\text{SL}(3, \mathbb{R})$ as we can see in the following

Example 2.2.8 Consider the map $\Phi: \mathrm{SL}(3, \mathbb{R}) \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ defined by

$$\Phi(A, p) = \frac{Ap}{\|Ap\|}. \quad (2.2.30)$$

The denominator of Eq. (2.2.30) is not zero since $A \in \mathrm{SL}(3, \mathbb{R})$ is invertible and $0 \notin \mathbb{S}^2$. Moreover, $\frac{Ap}{\|Ap\|} \in \mathbb{S}^2$ since $\|\frac{Ap}{\|Ap\|}\| = 1$. Thus Φ is well-defined. It is smooth since all involved mappings are smooth and it is a Lie group action because for all $x \in \mathbb{S}^2$ and $A, B \in \mathrm{SL}(3, \mathbb{R})$ one has

$$\Phi(\mathbb{1}, x) = \frac{x}{\|x\|} = x \text{ and } \Phi(A, \Phi(B, x)) = \Phi(A, \frac{Bx}{\|Bx\|}) = \frac{A \frac{Bx}{\|Bx\|}}{\|A \frac{Bx}{\|Bx\|}\|} = \frac{ABx}{\|ABx\|} = \Phi(AB, x).$$

If we restrict Φ to $\mathrm{SO}(3) \times \mathbb{S}^2$ the action reads $\Phi(A, p) = \frac{Ap}{\|Ap\|} = Ap$, since $\|Ap\| = 1$ as it was shown in the last example. Thus Φ is indeed an extension of the action in Example 2.2.7. This shows immediately the transitivity of Φ . It is also effective, since any $\mathbb{1} \neq A \in \mathrm{SL}(3, \mathbb{R})$ has an eigenvector $1 \neq \lambda \in \mathbb{C}$. Consequently, $\mathbb{S}^2 \cong \mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{R})_p$ for any $p \in \mathbb{S}^2$, where the stabilizer $\mathrm{SL}(3, \mathbb{R})_p$ has to be 6-dimensional since $\dim(\mathrm{SL}(3, \mathbb{R})) = 8$. Take for example $e_1 = (1 \ 0 \ 0)^T \in \mathbb{S}^2$. Then the corresponding stabilizer is

$$\mathrm{SL}(3, \mathbb{R})_{e_1} = \left\{ \begin{pmatrix} \frac{1}{\det(A)} & \alpha & \beta \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{pmatrix} \in M_{3 \times 3}(\mathbb{R}) \mid \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}), \alpha, \beta \in \mathbb{R} \right\}.$$

To prove this, consider an arbitrary matrix $B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R})$ and calculate

$\|Be_1\| = \sqrt{B_{11}^2 + B_{21}^2 + B_{31}^2}$. Then $\frac{Be_1}{\|Be_1\|} = e_1$ implies $B_{21} = B_{31} = 0$ and $\|Be_1\| = B_{11}$. The condition $\det(B) = 1$ forces $\begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ and $B_{11} = \frac{1}{\det(B)}$. Counting degrees of freedom one obtains $\dim(\mathrm{SL}(3, \mathbb{R})_{e_1}) = 6$. Then the diffeomorphism between $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{R})_{e_1}$ and \mathbb{S}^2 is

$$\beta_{e_1}: \mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{R})_{e_1} \ni B \cdot \mathrm{SL}(3, \mathbb{R})_{e_1} \mapsto \frac{Be_1}{\|Be_1\|} \in \mathbb{S}^2. \quad (2.2.31)$$

For the next example we remind the reader of the notion of *Minkowski space* which is common in special relativity. Consider the matrix

$$I_{1,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}). \quad (2.2.32)$$

The space \mathbb{R}^4 together with the *Lorentz metric*

$$\varphi_{1,3}: \mathbb{R}^4 \times \mathbb{R}^4 \ni ((x^0, \dots, x^3), (y^0, \dots, y^3)) \mapsto x^0 y^0 - \sum_{k=1}^3 x^k y^k \in \mathbb{R} \quad (2.2.33)$$

associated to the matrix (2.2.32) is said to be the Minkowski space. The corresponding quadratic form is

$$\Phi_{1,3}: \mathbb{R}^4 \ni (x^0, \dots, x^3) \mapsto (x^0)^2 - \sum_{k=1}^3 (x^k)^2 \in \mathbb{R}. \quad (2.2.34)$$

The set $O(1, 3, \mathbb{R})$ of isometries of the Lorentz metric $\varphi_{1,3}$ is said to be the **Lorentz group**. Thus $\Lambda \in O(1, 3, \mathbb{R})$ if $\varphi_{1,3}(\Lambda x, \Lambda y) = \varphi_{1,3}(x, y)$ for all $x, y \in \mathbb{R}^4$. The set of matrices $\Lambda \in O(1, 3, \mathbb{R})$ that fulfil $\det(\Lambda) = 1$ is further denoted by $SO(1, 3, \mathbb{R})$. One often writes $x = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix} \in \mathbb{R}^4$ to indicate the *time part* $x^0 \in \mathbb{R}$ and the *space part* $\vec{x} \in \mathbb{R}^3$ of the vector x .

Example 2.2.9 Consider the **proper orthochronous Lorentz group**

$$\mathcal{L}_3^{\uparrow,+} = \{\Lambda \in SO(1, 3, \mathbb{R}) \mid \Lambda_{11} \geq 1\}. \quad (2.2.35)$$

It is the connected component of the unit in $O(1, 3, \mathbb{R})$ and a 6-dimensional non-compact Lie group. For vectors $\vec{x} \in \mathbb{S}^2 \subseteq \mathbb{R}^3$ and matrices $\Lambda \in \mathcal{L}_3^{\uparrow,+}$ one defines a map $\Phi: \mathcal{L}_3^{\uparrow,+} \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ by

$$\Phi(\Lambda, \vec{x}) = \frac{\text{pr} \left(\Lambda \begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \right)}{\left\| \text{pr} \left(\Lambda \begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \right) \right\|}, \quad (2.2.36)$$

where pr is the projection on the space part of a vector, i.e. $\text{pr}((x^0 \ x^1 \ x^2 \ x^3)^T) = (x^1 \ x^2 \ x^3)^T$ for all $(x^0 \ x^1 \ x^2 \ x^3)^T \in \mathbb{R}^4$. The map Φ is obviously well-defined and smooth. For any $\vec{x} \in \mathbb{S}^2$ one has

$$\Phi(\mathbb{1}, \vec{x}) = \frac{\text{pr} \left(\mathbb{1} \begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \right)}{\left\| \text{pr} \left(\mathbb{1} \begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \right) \right\|} = \frac{\vec{x}}{\|\vec{x}\|} = \vec{x},$$

since $\|\vec{x}\| = 1$. Proving the action property and effectiveness of Φ is a bit more involved. We want to give another argument to prove that there is a transitive action of $\mathcal{L}_3^{\uparrow,+}$ on \mathbb{S}^2 : we calculate the Iwasawa decomposition (see Definition B.3.6) of $\mathcal{L}_3^{\uparrow,+}$ following [42, Section 4.5]. The Lie algebra corresponding to $\mathcal{L}_3^{\uparrow,+}$ is

$$\mathfrak{so}(1, 3) = \left\{ \begin{pmatrix} 0 & \vec{a}^T \\ \vec{a} & A \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}) \mid \vec{a} \in \mathbb{R}^3, A^T = -A \in M_{3 \times 3}(\mathbb{R}) \right\}, \quad (2.2.37)$$

what can be concluded quite easily since the Lie algebra of the special orthogonal matrices $SO(4)$ are the skew-symmetric matrices $\mathfrak{so}(4)$. This can be adapted to signature $(1, 3)$. Then the Cartan decomposition (see Definition B.2.7 of $\mathfrak{so}(1, 3)$) is

$$\mathfrak{so}(1, 3) = \mathfrak{k} \oplus \mathfrak{p}, \quad (2.2.38)$$

where

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}) \mid A = -A^T \in M_{3 \times 3}(\mathbb{R}) \right\} \quad (2.2.39)$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \vec{a}^T \\ \vec{a} & 0_{3 \times 3} \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}) \mid \vec{a} \in \mathbb{R}^3 \right\}. \quad (2.2.40)$$

This can be obtained as follows: consider the Cartan involution (see Definition B.2.5) $\Theta: \mathfrak{so}(1, 3) \ni B \mapsto -B^T \in \mathfrak{so}(1, 3)$ on $\mathfrak{so}(1, 3)$. Then

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}) \mid A = -A^T \in M_{3 \times 3}(\mathbb{R}) \right\} = \{B \in \mathfrak{so}(1, 3) \mid \Theta(B) = B\} = \mathfrak{k}$$

and

$$\left\{ \begin{pmatrix} 0 & \vec{a}^T \\ \vec{a} & 0_{3 \times 3} \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}) \mid \vec{a} \in \mathbb{R}^3 \right\} = \{B \in \mathfrak{so}(1, 3) \mid \Theta(B) = -B\} = \mathfrak{p}.$$

To get a maximal abelian subalgebra \mathfrak{a} of \mathfrak{p} one calculates the commutator of $A = \begin{pmatrix} 0 & \vec{a}^T \\ \vec{a} & 0_{3 \times 3} \end{pmatrix}$

and $B = \begin{pmatrix} 0 & \vec{b}^T \\ \vec{b} & 0_{3 \times 3} \end{pmatrix} \in \mathfrak{p}$:

$$[A, B] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a^1 b^2 - a^2 b^1 & a^1 b^3 - a^3 b^1 \\ 0 & a^2 b^1 - a^1 b^2 & 0 & a^2 b^3 - a^3 b^2 \\ 0 & a^3 b^1 - a^1 b^3 & a^3 b^2 - a^2 b^3 & 0 \end{pmatrix}.$$

Thus, it is easy to see that

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} \quad (2.2.41)$$

is a maximal abelian subalgebra of \mathfrak{p} . A nilpotent Lie subalgebra \mathfrak{n} of $\mathfrak{so}(1, 3)$ given as the sum of root spaces of a choice of positive roots on \mathfrak{a} is given by

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & b & c \\ b & -b & 0 & 0 \\ c & -c & 0 & 0 \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}) \mid b, c \in \mathbb{R} \right\}. \quad (2.2.42)$$

Thus the Iwasawa decomposition of $\mathfrak{so}(1, 3)$ is

$$\mathfrak{so}(1, 3) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \quad (2.2.43)$$

If we denote the corresponding connected Lie groups by K, A and N we get the Iwasawa decomposition of $\mathcal{L}_3^{\uparrow, +}$, namely

$$\mathcal{L}_3^{\uparrow, +} = KAN. \quad (2.2.44)$$

It is clear by definition that $K \cong \mathrm{SO}(3)$. Defining $M = \mathrm{SO}(2)$ we conclude by Example 2.2.7

$$\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2) \cong K/M \cong KAN/MAN \cong \mathcal{L}_3^{\uparrow,+}/MAN. \quad (2.2.45)$$

Since $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$ we know that $\mathfrak{a} \oplus \mathfrak{n}$ is a Lie subalgebra of $\mathfrak{so}(1, 3)$. Thus AN is a subgroup of $\mathcal{L}_3^{\uparrow,+}$ and we showed that \mathbb{S}^2 can be structured as a homogeneous space via a transitive action of $\mathcal{L}_3^{\uparrow,+}$.

We see in Section 2.4 that Example 2.2.7, Example 2.2.8 and Example 2.2.9 provide the only three (non-equivalent) ways to structure \mathbb{S}^2 as a homogeneous space via an effective action.

2.3 The Euler Characteristic of Compact Homogeneous Spaces

In Proposition 2.2.5 and Theorem 2.2.6 we gave a characterization of homogeneous spaces by quotient spaces of Lie groups that act transitively on the homogeneous space modulo a stabilizer of the action. There is another way to detect homogeneous spaces with a topological background.

For $k \in \mathbb{N}_0$ we denote the set of smooth k -differential forms on a smooth manifold M by $\Omega^k(M)$. An element $\omega \in \Omega^k(M)$ is said to be *closed* if the exterior derivative of ω is zero, i.e. $d\omega = 0$. It is called *exact* if there is a $(k-1)$ -form $\alpha \in \Omega^{k-1}(M)$ such that $\omega = d\alpha$. Since $d^2 = 0$ every exact k -form is closed and one can define the quotient space $H_{\mathrm{dR}}^k(M)$ of all closed k -forms modulo all exact k -forms which is called the k -th *de Rham cohomology* of M . From a Theorem by de Rham this is isomorphic to the k -th singular cohomology on M with coefficients in \mathbb{R} . The direct sum of all these spaces

$$H_{\mathrm{dR}}^\bullet(M) = \bigoplus_{k \in \mathbb{N}_0} H_{\mathrm{dR}}^k(M) \quad (2.3.1)$$

is said to be the de Rham cohomology of M .

Definition 2.3.1 (Euler characteristic) *The Euler characteristic $\chi(M)$ of a smooth manifold M is defined as*

$$\chi(M) = \sum_{k=0}^{\infty} (-1)^k \dim(H_{\mathrm{dR}}^k(M)). \quad (2.3.2)$$

Remark 2.3.2 In Definition 2.3.1 one has to argue that the series (2.3.2) converges. While this is true for compact smooth manifolds this will not be the case for arbitrary smooth manifolds. However we are mainly interested in compact manifolds M , that also fulfil $\dim(H_{\mathrm{dR}}^k(M)) < \infty$ for all $k \in \mathbb{N}_0$. For more informations about the Euler characteristic of a smooth manifold consider [21, Chapter 11].

This is our tool to detect homogeneous spaces. A homogeneous space M is compact if for example the Lie group G that acts transitively on M is compact. We prove that in this situation the Euler characteristic of M is non-negative, following [93, Section 5]. We need to sum up some basic definitions and properties of Lie group representations first.

Definition 2.3.3 (Lie Group Representation) *A representation of a Lie group G is a finite-dimensional real vector space V together with a Lie group homomorphism*

$$\pi: G \rightarrow \mathrm{GL}(V), \quad (2.3.3)$$

where $\mathrm{GL}(V)$ denotes the general linear group of V consisting of the bijective linear maps $V \rightarrow V$ with concatenation as group multiplication.

Example 2.3.4 Consider the **adjoint representation** $(\text{Ad}, \mathfrak{g})$ of G on \mathfrak{g} , where

$$\text{Ad}: G \ni g \mapsto T_e \text{Conj}_g \in \text{GL}(\mathfrak{g}). \quad (2.3.4)$$

The map $\text{Ad}^*: G \rightarrow \text{GL}(\mathfrak{g}^*)$ that assigns an element $g \in G$ the adjoint of the linear map $\text{Ad}(g^{-1})$ is said to be the **coadjoint representation** of G on \mathfrak{g}^* . For any $k \in \mathbb{N}$ one can extend those two maps linearly to obtain representations $\Lambda^k \text{Ad}$ and $\Lambda^k \text{Ad}^*$ of G on $\Lambda^k \mathfrak{g}$ and $\Lambda^k \mathfrak{g}^*$, respectively.

Now choose an arbitrary representation (π, V) of G . If there is a scalar product $\langle \cdot, \cdot \rangle$ on V such that

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \quad (2.3.5)$$

for all $g \in G$ and $v, w \in V$ the representation (π, V) is said to be *orthogonal*. A real vector space can always be endowed with a scalar product. We want to prove that there is a scalar product on V satisfying (2.3.5) for a chosen representation (π, V) if G is compact. The first step is to consider a special measure. Take a n -dimensional manifold M and a *volume form* ω on it, i.e. $\omega = w dx_1 \wedge \dots \wedge dx_n$ in local coordinates (U, x_i) . The *measure* μ corresponding to ω is then defined by the relation

$$\int f d\mu = \int_U f(x_1, \dots, x_n) |w| dx_1 \cdots dx_n \quad (2.3.6)$$

for all $f \in \mathcal{C}^\infty(U)$. One can check that this is independent of the chart (U, x_i) . In the case of a Lie group G we choose a non-zero element $\tilde{\omega} \in \Lambda^n(T_e G)$ and define $\omega = T_e L_g \tilde{\omega}$ for any $g \in G$. If G is compact in addition we can normalize ω such that the corresponding measure fulfils

$$\int_G dg = 1. \quad (2.3.7)$$

It has been proved in [45] that this measure is unique and bi-invariant, i.e.

$$\int_G f(g) dg = \int_G f(gh) dg = \int_G f(hg) dg \quad (2.3.8)$$

for all $f \in \mathcal{C}^\infty(G)$ and $h \in G$. We call it the *normalized Haar measure* on G . Now we come back to orthogonal representations:

Proposition 2.3.5 *Let (π, V) be a representation of a compact Lie group G and $\langle \cdot, \cdot \rangle'$ any scalar product on V . Then*

$$\langle h_1, h_2 \rangle = \int_G \langle \pi(g)h_1, \pi(g)h_2 \rangle' dg \quad (2.3.9)$$

is a scalar product on V such that Eq. (2.3.5) holds, i.e. (π, V) is a orthogonal representation of G .

PROOF: Clearly Eq. (2.3.9) defines a scalar product on G . It fulfils (2.3.5) since for $g', h_1, h_2 \in G$ one has

$$\begin{aligned} \langle \pi(g')h_1, \pi(g')h_2 \rangle &= \int_G \langle \pi(g)\pi(g')h_1, \pi(g)\pi(g')h_2 \rangle' dg \\ &= \int_G \langle \pi(gg')h_1, \pi(gg')h_2 \rangle' dg \\ &= \int_G \langle \pi(g)h_1, \pi(g)h_2 \rangle' dg \\ &= \langle h_1, h_2 \rangle. \end{aligned} \quad \square$$

Also remark the notion of maximal tori:

Definition 2.3.6 *Let G be a Lie group. Then a Lie subgroup T is said to be a*

- i.) **torus** in G if T is isomorphic to a product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ of 1-spheres.*
- ii.) **maximal torus** in G if for any torus T' in G such that $T \subseteq T'$ one has $T' = T$.*

These objects are of fundamental use in the theory of connected compact Lie groups. Some properties are collected in the following

Proposition 2.3.7 *Let G be a connected compact Lie group.*

- i.) Any torus in G is contained in a maximal torus in G .*
- ii.) Any element in G is contained in a maximal torus in G .*
- iii.) Any two maximal tori in G are conjugated to each other, i.e. if T and T' are maximal tori in G there is a $g \in G$ such that*

$$gTg^{-1} = T'. \quad (2.3.10)$$

*For this one calls the dimension of any maximal torus in G the **rank** of G .*

- iv.) Any maximal torus T in G has a **generating element** $g_0 \in T$, i.e. for any $g \in T$ there is a natural number $k \in \mathbb{N}_0$ such that $g = g_0^k$.*

All statements are taken from [3, Section 2.4], [23, Chapter IV] and [53, Chapter 6]. Consider those references for more informations about maximal tori.

Furthermore, one has a topological result on representations of compact Lie groups. In analogy to Definition 2.3.16 we want to define the Euler characteristic of a representation. Let (π, V) be a representation of a compact Lie group G with $n = \dim(V)$. Then for any $1 \leq k \leq n$ the map π induces a representation $\Lambda^k \pi$ on $\Lambda^k V$. Denote by I_k the subspaces of $\Lambda^k V$ consisting of the elements v that satisfy $(\Lambda^k \pi(g))(v) = v$ for all $g \in G$. Then define the *Poincaré polynomial* P_π of (π, V) by

$$P_\pi(t) = \sum_{k=1}^n \lambda_k t^k, \quad (2.3.11)$$

where $\lambda_k = \dim(I_k)$ are the *Betti numbers* corresponding to π . We know from representation theory (c.f. [28, Section 12.6.2]) that the Betti numbers are obtained by integration of the trace of the corresponding representation over the whole group, i.e.

$$\lambda_k = \int_G \text{tr}(\Lambda^k \pi(g)) dg. \quad (2.3.12)$$

Definition 2.3.8 *Let (π, V) be a representation of a compact Lie group G . The Euler characteristic $\chi(\pi)$ of (π, V) is defined by*

$$\chi(\pi) = P_\pi(-1). \quad (2.3.13)$$

In the end we want to prove that the Euler characteristic of a connected compact homogeneous space is non-negative. The next lemma is a step towards this:

Lemma 2.3.9 *Let (π, V) be a representation of a connected compact Lie group G . Then $\chi(\pi) \geq 0$. Moreover, one has $\chi(\pi) > 0$ if and only if for each maximal torus T in G one has $\pi(g)v = v$ for all $g \in T$, only for a $v \in T$ if $v = 0$.*

PROOF: The discussion above implies

$$\begin{aligned}\chi(\pi) &= P_\pi(-1) = \sum_{k=1}^n \lambda_k (-1)^k = \sum_{k=1}^n \int_G \text{tr}(\Lambda^k \pi(g)) dg (-1)^k \\ &= \int_G \sum_{k=1}^n \text{tr}(\Lambda^k \pi(g)) (-1)^k dg \stackrel{(*)}{=} \int_G \det(-\pi(g) + \text{id}_V) dg,\end{aligned}$$

where in $(*)$ we used the formula

$$\sum_{k=1}^n \text{tr}(\Lambda^k \pi(g)) t^k = \det(t\pi(g) + \text{id}_V) \quad (2.3.14)$$

resulting from a principal minor argument. According to Proposition 2.3.5 we can assume that $\pi(g)$ is a orthogonal transformation for all $g \in G$. Since G is connected $\pi(g)$ is also orientation preserving. First assume that n is odd. Then according to [43, Lemma 7.3.1] $\pi(g)$ has an eigenvalue $\lambda = 1$, i.e. $\det(-\pi(g) + \text{id}_V) = 0$. Thus in this case $\chi(\pi) = 0$. If n is even one has the normal form of orthogonal matrices

$$\begin{pmatrix} R_1 & & & & \\ & \ddots & & & \\ & & R_m & & \\ & & & \pm 1 & \\ & & & & \ddots \\ & & & & & \pm 1 \end{pmatrix}, \quad (2.3.15)$$

with $R_i = \begin{pmatrix} \cos(\phi_i) & -\sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) \end{pmatrix}$ for some $\phi_i \in \mathbb{R}$. If at least one -1 appears in the matrix (2.3.15) one has $\det(-\pi(g) + \text{id}_V) = 0$. Since $\det(R_i) = 1$ and $\pi(g)$ preserves orientation there is a even number of 1 appearing in (2.3.15) and since we have

$$\begin{aligned}\det \left(- \begin{pmatrix} \cos(\phi_i) - 1 & -\sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) - 1 \end{pmatrix} \right) &= (-1)^2 (\cos^2(\phi_i) - 2 \cos(\phi_i) + 1 + \sin^2(\phi_i)) \\ &= 2(1 - \cos(\phi_i)) \geq 0,\end{aligned}$$

this implies $\det(-\pi(g) + \text{id}_V) \geq 0$. Thus $\chi(\pi) \geq 0$ and we have proven the first claim of the proposition. We prove both directions of the second statement by contraposition. Assume $\chi(\pi) = 0$. Since $\det(-\pi(g) + \text{id}_V) \geq 0$, this implies $\det(-\pi(g) + \text{id}_V) = 0$ for all $g \in G$. Choose any maximal torus T in G with generating element $g_0 \in G$. Then of course $\det(-\pi(g_0) + \text{id}_V) = 0$ which means that 1 is an eigenvalue of $\pi(g_0)$, i.e. $\pi(g_0)$ fixes a non-zero vector $v \in V$. Since π is a Lie group homomorphism and any other element g in T is obtained as a power $p \in \mathbb{N}$ of g_0 one has $\pi(g)v = \pi(g_0^p)v = \pi(g_0) \cdots \pi(g_0)v = v$ and every element of T fixes v . Conversely, assume that there is a maximal torus T fixing a non-zero vector $v \in V$, i.e. $\det(-\pi(g) + \text{id}_V) = 0$ for all $g \in T$. Now any element $g \in G$ is conjugated to an element $g_0 \in T$, i.e. there is an $h \in G$ such that $g = hg_0h^{-1}$. Via cyclic permutations of the arguments of the determinant this gives together with the homomorphism property of π

$$\det(-\pi(g) + \text{id}_V) = \det(-\pi(h^{-1})\pi(h)\pi(g_0) + \text{id}_V) = 0. \quad (2.3.16)$$

This proves $\det(-\pi(g) + \text{id}_V) = 0$ for all $g \in G$ and $\chi(\pi) = 0$. \square

The next proof deals with differential forms on a manifold that are invariant under the action of a Lie group that acts on the manifold. We denote by α a differential k -form on a smooth manifold M and we say that it is *invariant* under the action Φ of a Lie group G on M if

$$\Phi_g^* \alpha = \alpha \text{ for all } g \in G, \quad (2.3.17)$$

i.e. for all $p \in M$, $g \in G$ and all tangent vectors $X_1, \dots, X_k \in T_p M$ one has

$$\alpha_{\Phi(g,p)}(T_p \Phi_g X_1, \dots, T_p \Phi_g X_k) = \alpha_p(X_1, \dots, X_k). \quad (2.3.18)$$

We use the notation

$$\Omega^k(M)^{G,\Phi} = \left\{ \alpha \in \Omega^k(M) \mid \Phi_g^* \alpha = \alpha \text{ for all } g \in G \right\} \quad (2.3.19)$$

to denote set of k -forms on M that are invariant under the action Φ . The corresponding k -th *invariant de Rham cohomology* is then defined by

$$H_{\text{dR}}^k(M)^{G,\Phi} = \frac{\{\alpha \in \Omega^k(M)^{G,\Phi} \mid d\alpha = 0\}}{\{d\beta \in \Omega^k(M) \mid \beta \in \Omega^{k-1}(M)^{G,\Phi}\}}, \quad (2.3.20)$$

and the direct sum of all these spaces is the invariant de Rham cohomology $H_{\text{dR}}^\bullet(M)^{G,\Phi}$. It is interesting to compare it to $H_{\text{dR}}^\bullet(M)$.

Lemma 2.3.10 *Let G be a connected compact Lie group that acts on a connected manifold M via Φ . Then*

$$H_{\text{dR}}^\bullet(M) \cong H_{\text{dR}}^\bullet(M)^{G,\Phi}. \quad (2.3.21)$$

PROOF: Choose an element $g \in G$. Since G is connected there is a smooth curve $\gamma: [0, 1] \rightarrow G$ in G such that $\gamma(0) = e$ and $\gamma(1) = g$. Furthermore one defines the smooth map

$$\Psi_g: [0, 1] \times M \ni (t, p) \mapsto \Psi_g(t, p) = \Phi_{\gamma(t)}(p) \in M. \quad (2.3.22)$$

Since $\Psi_g(0, \cdot) = \Phi_{\gamma(0)} = \text{id}_M$ and $\Psi_g(1, \cdot) = \Phi_{\gamma(1)} = \Phi_g$, the map Ψ_g is a homotopy of id_M and Φ_g . By a homotopy argument (see [108, Satz 2.2.1]) the two maps

$$\text{id}_M^* = \Phi_g^*: H_{\text{dR}}^\bullet(M) \rightarrow H_{\text{dR}}^\bullet(M) \quad (2.3.23)$$

coincide on cohomology level. This construction is valid for all $g \in G$. In a next step we consider the map

$$\Xi: \Omega^\bullet(M) \ni \alpha \mapsto \int_G \Phi_g^* \alpha dg \in \Omega^\bullet(M)^{G,\Phi}. \quad (2.3.24)$$

It is well-defined, since

$$\Phi_h^* \int_G \Phi_g^* \alpha dg = \int_G \Phi_{gh}^* \alpha dg = \int_G \Phi_g^* \alpha dg,$$

for any $\alpha \in \Omega^\bullet(M)$, by the bi-invariance of the normalized Haar measure. We want to show that

$$\hat{\Xi}: H_{\text{dR}}^\bullet(M) \ni [\alpha] \mapsto [\Xi(\alpha)] \in H_{\text{dR}}^\bullet(M)^{G,\Phi} \quad (2.3.25)$$

defines a linear isomorphism. The map $\hat{\Xi}$ is well-defined since $\Xi(\alpha) \in \Omega^\bullet(M)^{G,\Phi}$ for any $\alpha \in \Omega^\bullet(M)$. Thus $\Xi(\alpha)$ is a representative of the equivalence class $\hat{\Xi}([\alpha])$ in $H_{\text{dR}}^\bullet(M)^{G,\Phi}$. The linearity of $\hat{\Xi}$ is clear. Let $[\alpha] \in H_{\text{dR}}^\bullet(M)^{G,\Phi} \subseteq H_{\text{dR}}^\bullet(M)$ be arbitrary. Then

$$[\alpha] = \left[\int_G \Phi_g^* \alpha dg \right] = [\Xi(\alpha)],$$

which implies that $\hat{\Xi}$ is surjective. By Eq. (2.3.23) one has for any $[\alpha] \in H_{\text{dR}}^\bullet(M)$

$$\begin{aligned} [\alpha] &= \left[\int_G \text{id}_M^* \alpha dg \right] \\ &= \left[\int_G \Phi_g^* \alpha dg \right] \\ &= [\Xi(\alpha)], \end{aligned}$$

i.e. $\hat{\Xi}$ is injective. This concludes the proof. \square

Now consider a connected homogeneous space $M = G/H$, where G is a connected and compact Lie group. Set $e' = e \cdot H$. One can define a left H -module structure on $\Lambda^\bullet T_{e'}^* M$ by

$$H \times \Lambda^\bullet T_{e'}^* M \ni (h, \alpha_{e'}) \mapsto h \triangleright \alpha_{e'} = \Lambda^\bullet \text{Ad}_h^*(\alpha_{e'}) \in \Lambda^\bullet T_{e'}^* M. \quad (2.3.26)$$

We denote the elements that are invariant under this action by

$$(\Lambda^\bullet T_{e'}^* M)^H = \{ \alpha_{e'} \in \Lambda^\bullet T_{e'}^* M \mid \Lambda^\bullet \text{Ad}_h^*(\alpha_{e'}) = \alpha_{e'} \text{ for all } h \in H \}. \quad (2.3.27)$$

Then one can prove the following

Lemma 2.3.11 *Let $M = G/H$ be a connected homogeneous space such that G is connected and compact, where we denote the transitive action of G on M by Φ . Then*

$$\Omega^\bullet(M)^{G,\Phi} \cong (\Lambda^\bullet T_{e'}^* M)^H. \quad (2.3.28)$$

PROOF: Consider the map

$$F: (\Lambda^\bullet T_{e'}^* M)^H \ni \alpha_{e'} \mapsto (M \ni p \mapsto \Phi_g^* \alpha_{e'} \in \Lambda^\bullet T_p^* M, \text{ where } p = \Phi_g(e')) \in \Omega^\bullet(M)^{G,\Phi}. \quad (2.3.29)$$

It is easy to see that $F(\alpha_{e'}) \in \Omega^\bullet(M)^{G,\Phi}$ for any $\alpha_{e'} \in (\Lambda^\bullet T_{e'}^* M)^H$. To prove that this map is well-defined one has to argue that $\Phi_g(e') = \Phi_h(e')$, for any $g, h \in G$ and $\alpha_{e'} \in (\Lambda^\bullet T_{e'}^* M)^H$, implies $\Phi_g^* \alpha_{e'} = \Phi_h^* \alpha_{e'}$ or equivalently $(\Phi_g \circ \Phi_{h^{-1}})^* \alpha_{e'} = \alpha_{e'}$. Since $\Phi_g(e') = \Phi_h(e')$ for $g, h \in G$ implies that $g^{-1}h \in H$, it suffices to prove that $\Phi_h^* \alpha_{e'} = \alpha_{e'}$ for all $h \in H$ and $\alpha_{e'} \in (\Lambda^\bullet T_{e'}^* M)^H$. Since Φ is transitive it is also locally transitive, i.e. the corresponding Lie algebra action $\phi|_p: \mathfrak{g} \rightarrow T_p M$ is surjective for all $p \in M$. Thus, for any $v_p \in T_p M$ there is a $\xi \in \mathfrak{g}$ such that the fundamental vector field ξ_M for ξ on M satisfies

$$\xi_M(p) = T_e \Phi_p \xi = v_p, \quad (2.3.30)$$

for any $p \in M$. Applying this to every tensor component of an arbitrary $X_p \in \Lambda^k T_p M$, one obtains an element $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \Lambda^k \mathfrak{g}$ such that

$$T_e \Phi_p \xi := T_e \Phi_p \xi_1 \wedge \cdots \wedge T_e \Phi_p \xi_k = X_p. \quad (2.3.31)$$

We denote the left hand side of Eq. (2.3.31) also by ξ_M . Let $\alpha_{e'} \in (\Lambda^k T_{e'}^* M)^H$, $X_{e'} \in \Lambda^k T_{e'} M$ and $h \in H$ be arbitrary. Then

$$\begin{aligned} (\Phi_h^* \alpha_{e'})(X_{e'}) &= \alpha_{e'}(T_{e'} \Phi_h X_{e'}) = \langle \alpha_{e'}, T_{e'} \Phi_h \xi_M(e') \rangle = \langle \alpha_{e'}, \Phi_{h^{-1}}^* \xi_M(e') \rangle \\ &= \langle \alpha_{e'}, \text{Ad}_u(\xi_M(e')) \rangle = \langle \text{Ad}_u^* \alpha_{e'}, X_{e'} \rangle = \langle \alpha_{e'}, X_{e'} \rangle, \end{aligned}$$

where we used fundamental equations of the Cartan calculus. This shows $\Phi_h^* \alpha_{e'} = \alpha_{e'}$ for all $h \in H$, i.e. F is well-defined. It is clear that F is linear. To show that it is injective choose $\alpha_{e'}, \beta_{e'} \in (\Lambda^k T_{e'}^* M)^H$ and assume $F(\alpha_{e'}) = F(\beta_{e'})$. Since $e' = \Phi_e(e')$ this implies

$$\alpha_{e'} = F(\alpha_{e'})(e) = F(\beta_{e'})(e) = \beta_{e'}.$$

To prove that F is surjective choose an arbitrary $\alpha \in \Omega^k(M)^{G,\Phi}$, i.e. $\Phi_g^* \alpha = \alpha$ for all $g \in G$. Let $X_{e'} \in \Lambda^k T_{e'} M$. Then for any $g \in G$ one has

$$\alpha_{e'}(X_{e'}) = \Phi_g^* \alpha_{e'}(X_{e'}) = \alpha_{\Phi_g(e')}(\Lambda^k T_{e'} \Phi_g X_{e'})$$

and since $T_{e'} \Phi_g$ is a linear isomorphism this implies for any $p \in M$ that

$$\alpha_p(X_p) = \alpha_{e'}(\Lambda^k T_p \Phi_{g^{-1}} X_p),$$

where $g \in G$ such that $\Phi_g(e') = p$. This concludes the proof. \square

Now we can prove that the Euler characteristic of a connected compact homogeneous space is non-negative if there is a connected compact Lie group acting transitively on it.

Proposition 2.3.12 *Let $M = G/H$ be a connected compact homogeneous space such that G is a connected compact Lie group and H a connected subgroup of G . Then $\chi(M) \geq 0$. Moreover, $\chi(M) > 0$ if and only if H is a subgroup of maximal rank, i.e. if any maximal torus in H has the same dimension as H .*

PROOF: Parts of the proof are taken from [93, Theorem II]. The exterior derivative d makes $(\Lambda^\bullet T_{e'}^* M)^H$ into a complex. This is a consequence of $d\alpha \in (\Lambda^{k+1} T_{e'}^* M)^H$ for $\alpha \in (\Lambda^k T_{e'}^* M)^H$, which is true since d and the pull back of any function commute. Any k -cochain of this complex can be identified with a k -cochain on the complex $(\Omega^\bullet(M)^{G,\Phi}, d)$ according to Lemma 2.3.11. Any k -cochain of $\Omega^\bullet(M)^{G,\Phi}$ can be written as the direct sum of an invariant $(k+1)$ -coboundary and an invariant k -cocycle. Moreover, any invariant k -cocycle can be written as the direct sum of an invariant k -coboundary and an element of the k -th invariant de Rham cohomology. Altogether, this implies

$$C^k(M)^{G,\Phi} \cong B^{k+1}(M)^{G,\Phi} \oplus H_{\text{dR}}^k(M)^{G,\Phi} \oplus B^k(M)^{G,\Phi}, \quad (2.3.32)$$

where $C^k(M)^{G,\Phi}$ and $B^k(M)^{G,\Phi}$ denote the set of all invariant k -cocycles and k -coboundaries on M , respectively. Then it follows from Lemma 2.3.10 that

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \dim((\Lambda^k T_{e'}^* M)^H) &= \sum_{k=0}^{\infty} (-1)^k \dim(C^k(M)^{G,\Phi}) \\ &\stackrel{(*)}{=} \sum_{k=0}^{\infty} (-1)^k \dim(H_{\text{dR}}^k(M)^{G,\Phi}) \\ &= \sum_{k=0}^{\infty} (-1)^k \dim(H_{\text{dR}}^k(M)) \end{aligned}$$

$$= \chi(M),$$

where we used Eq. (2.3.32) in (*). But since there is the representation $\Lambda^\bullet \text{Ad}^*$ of the connected compact Lie group H on $\Lambda^\bullet T_e^* M$ Lemma 2.3.9 implies

$$0 \leq \chi(\Lambda^\bullet \text{Ad}^*) = \sum_{k=0}^{\infty} (-1)^k \dim((\Lambda^k T_e^* M)^H) = \chi(M). \quad (2.3.33)$$

Also the statement about the maximal torus can be deduced from Lemma 2.3.9. \square

The next step consists in extending this result to connected non-compact Lie groups. This is in general very hard to obtain. The situation gets easier if one assumes the space M to be simply connected in addition, what will be enough for considering the sphere \mathbb{S}^2 as an example. Nevertheless, we cite the proofs for the case when M is only assumed to be connected since we are also interested in the pretzel surfaces that are not simply connected. We want to get a tool to use Proposition 2.3.12 also in the case of non-compact Lie groups G if M is simply connected. If G is connected and M simply connected, any stabilizer G_x , where $x \in M$, of the transitive action of G on M has to be connected since $M = G/G_x$. This means we can use the following famous theorem of Cartan for each, G and G_x (c.f. [56, Theorem 2]):

Theorem 2.3.13 *Every connected Lie group G is a Cartesian product of a maximal compact Lie subgroup $K \subseteq G$ and an Euclidean vector space E , i.e.*

$$G = K \times E. \quad (2.3.34)$$

Assume that $M = G/G_x$ is simply connected. Then there are compact Lie subgroups K, L and Euclidean vector spaces E, F such that $G = K \times E$ and $G_x = L \times F$ according to Theorem 2.3.13. Since $G_x \subseteq G$ and any two maximal compact subsets of G are conjugate to each other we can assume $L \subseteq K$. Now G/L can be seen as the total space of a vector bundle once with basis G/K and fibre K/L and twice with basis G/G_x and fibre G_x/L . For example $\pi: G/L \rightarrow G/K$ maps each coset of L in G to the coset of K in G which contains it. But since $G/K \cong E$ is an Euclidean space, a general topological result says that this vector bundle is trivial, i.e.

$$G/L = K/L \times E. \quad (2.3.35)$$

The same holds for the second vector bundle, which implies that there is a global section

$$f: M \cong G/G_x \rightarrow G/L. \quad (2.3.36)$$

Then some topological estimations (see [73, Corollary 1]) imply the following

Theorem 2.3.14 *If $M = G/H$ is a simply connected compact homogeneous space and G a connected Lie group (probably non-compact) there exists a connected compact Lie subgroup K of G that acts transitively on M , i.e.*

$$M = G/H = K/L, \quad (2.3.37)$$

where $L = K \cap H$.

An immediate consequence of the last theorem and Proposition 2.3.12 is

Theorem 2.3.15 *Let $M = G/H$ be a simply connected compact homogeneous space. Then $\chi(M) \geq 0$. One has $\chi(M) > 0$ if and only if H has full rank.*

There is an even more general result due to A. L. Onishchik and V. V. Gorbatsevich which shows that the assumption of M being simply connected is not necessary (c.f. [83, Part II, Chapter 5, Theorem 1.2]).

Theorem 2.3.16 *Let $M = G/H$ be a connected compact homogeneous space. Then $\chi(M) \geq 0$. One has $\chi(M) > 0$ if and only if H has full rank.*

The proof is proposing Tits bundles. Remark that there is a different proof of this theorem from 2005 of G. D. Mostow that uses iterated fibrations (c.f. [76]).

2.4 The Classification of Transitive Actions on \mathbb{S}^2

The goal of this section is to describe all ways one can structure the 2-sphere as a homogeneous space. As we discovered in Section 2.2 it is equivalent to determine all transitive actions on \mathbb{S}^2 . This classification has to be done with a notion of equivalent actions.

Definition 2.4.1 *Assume there are two Lie groups G and G' acting on a manifold M via Φ and Φ' , respectively. These two actions are said to be **equivalent** if there is a Lie group morphism $\phi: G \rightarrow G'$ such that*

$$\Phi(g, x) = \Phi'(\phi(g), x) \quad (2.4.1)$$

*for all $g \in G$ and $x \in M$. They are called **locally equivalent** if there is an open subset $U \subseteq M$ such that $\Phi|_{G \times U}$ and $\Phi'|_{G \times U}$ are equivalent.*

It is clear by definition that equivalent actions are locally equivalent. It is also quite intuitive that a classification makes only sense if one forces the actions to be effective. Otherwise one could enlarge the Lie group by adding elements that simply induce the identity diffeomorphism on the manifold. It is reasonable to classify transitive Lie groups such that every element induces a different diffeomorphism, i.e. actions that are effective in addition. In Chapter 6 we realize that this assumption is no restriction at all for our purpose. In the 1960's many classifications of transitive effective actions appeared. Pioneer work was done by A.L. Onishchik. He considered compact homogeneous spaces and gave classifications of compact Lie groups that act as well as for non-compact Lie groups (see e.g. [80–83]). In [83, Theorem 2.6] he classified in particular all locally effective transitive actions of connected compact Lie groups on simply connected homogeneous spaces of rank 1. We follow a source Onishchik also uses, namely the paper [74] of D. Montgomery and H. Samelson. There they classify all connected compact Lie groups that act transitively and effectively on \mathbb{S}^n , $n \in \mathbb{N}$. We want to mention that the classification splits into even and odd n . But since we are just interested in $n = 2$ we do not see these effects. We first prove that if G acts in the way described above it already has to be simple. Then there is just a list of Lie groups left to check. For this we need a little

Lemma 2.4.2 *Let G_1, G_2 be connected compact Lie groups and N a finite normal subgroup of $\overline{G} = G_1 \times G_2$. If $G = (G_1 \times G_2)/N$ acts transitively on \mathbb{S}^2 then for one $i = 1, 2$ and $x \in \mathbb{S}^2$ the Lie group $G_i/(G_i \cap G_x)$ acts transitively on \mathbb{S}^2 .*

PROOF: The proof and the notation is taken from [74, Theorem I']. If G acts transitively on \mathbb{S}^2 then of course also \overline{G} acts transitively on it. Take the stabilizer \overline{G}_x of this action of any point $x \in \mathbb{S}^2$. It is connected because \mathbb{S}^2 is simply connected and compact. Moreover, any element $h \in \overline{G}_x$ can be written as $h = h_1 \times h_2$ with some $h_1 \in G_1$ and $h_2 \in G_2$ and is contained in a maximal toral subgroup $T \subseteq \overline{G}_x$. It is proven in [92] that \overline{G} and \overline{G}_x have the same rank, which implies that $T \subseteq \overline{G}$, thus $T = T_1 \times T_2$, where T_1 and T_2 are maximal toral subgroups of G_1 and G_2 , respectively. This implies that $h_1, h_2 \in T \subseteq \overline{G}_x$, i.e. $\overline{G}_x = H_1 \times H_2$ with $H_i = G_i \cap \overline{G}_x$. Thus, we have a decomposition

$$\overline{G}/\overline{G}_x \cong G_1/H_1 \times G_2/H_2 \quad (2.4.2)$$

and $G_1/H_1 \times G_2/H_2 \cong \overline{G}/\overline{G}_x \cong \mathbb{S}^2$. But in this case it is known that G_1/H_1 or G_2/H_2 has to be a single point. Let $G_2/H_2 \cong \{\text{pt}\}$, i.e. $G_2 = H_2$ which means that the elements of G_2 stabilize the point x in the action of \overline{G} on \mathbb{S}^2 . Then G_1/H_1 has to be transitive on \mathbb{S}^2 . \square

Now, following [74, Theorem I], we have

Theorem 2.4.3 *Every connected compact Lie group that acts transitively and effectively on \mathbb{S}^2 is simple.*

PROOF: Assume that the connected compact Lie group G acts transitively and effectively on \mathbb{S}^2 but is not simple. Since G is a connected and compact Lie group there are two connected compact subgroups G_1 and G_2 of G and a finite normal subgroup N of $\overline{G} = G_1 \times G_2$ such that $G = (G_1 \times G_2)/N$ according to [53, Theorem 6.19]. Since G acts effectively we know that \overline{G} has to act almost effectively, i.e. the action of \overline{G} on \mathbb{S}^2 only has a finite set of fixed points, namely the elements of N . This implies that \overline{G}_x can not contain any infinite normal subgroup of \overline{G} for any $x \in \mathbb{S}^2$. But as in the proof of Lemma 2.4.2 we conclude that $G_2 \subseteq \overline{G}_x$ stabilizes the action of \overline{G} . Now G_2 is a non-finite normal subgroup which gives the contradiction. \square

Theorem 2.4.4 *Any effective transitive action of a connected compact Lie group on \mathbb{S}^2 is equivalent to the linear action of $\text{SO}(3)$ on \mathbb{S}^2 .*

The proof of Theorem 2.4.4 can be summarized as follows: according to Theorem 2.4.3 any connected compact Lie group G that acts transitively and effectively on \mathbb{S}^2 has to be simple. By the Cartan-Killing classification there is a finite list of all connected compact simple Lie groups that are not locally isomorphic (c.f. [20]). In [92] the homology rings of these simple Lie groups are connected to the homology rings of Cartesian products of spheres of several dimensions. It comes out that for dimensional reasons the only simple Lie group satisfying these conditions is $\text{SO}(3)$ with stabilizer $\text{SO}(2)$ (c.f. [74, Theorem II' and Lemma 1]). Remark that the action of $\text{SO}(3)$ on \mathbb{S}^2 is the one we discussed in Example 2.2.7. In a very similar way one proves the following result for connected non-compact Lie groups:

Lemma 2.4.5 *Any connected Lie group that acts transitively and locally effectively on \mathbb{S}^2 is semisimple.*

Again we just want to sketch the proof: Onishchik proves this theorem for all compact homogeneous spaces with positive Euler characteristic and also gives a decomposition of the Lie group into simple normal divisors (c.f. [82, Theorem 2]). The proof bases on the classification of the \mathfrak{t} -subalgebras and \mathfrak{k} -subalgebras of all semisimple Lie groups which was also done by Onishchik in [81]. In this paper he constructs standard \mathfrak{t} - and \mathfrak{k} -subgroups and shows that any

\mathfrak{t} - and \mathfrak{k} -subgroup of a semisimple Lie group is conjugate to a standard one on Lie algebra level (c.f. [81, Theorem 3 and Theorem 4]).

Since semisimple Lie groups are well known and classified one can conclude the next theorem (c.f. [82, Theorem 3]).

Theorem 2.4.6 *Any connected Lie group that acts transitively and effectively on \mathbb{S}^2 is equivalent to $\mathrm{SO}(3)$, $\mathrm{SL}(3, \mathbb{R})$ or $\mathcal{L}_3^{\uparrow,+}$.*

Remark that the actions mentioned in Theorem 2.4.6 are given in Example 2.2.7, Example 2.2.8 and Example 2.2.9, respectively. This is also consistent with Theorem 2.3.14 since $\mathrm{SO}(3)$ is a compact subgroup of $\mathcal{L}_3^{\uparrow,+}$ and $\mathrm{SL}(3, \mathbb{R})$ that acts transitively on \mathbb{S}^2 .

Chapter 3

Lie Bialgebras and r -Matrices

We discussed the geometry of homogeneous spaces and their connection to transitive Lie group actions. In a way, it is natural to study those Lie groups in order to learn more about homogeneous spaces. In Subsection 2.4 we showed that for some examples only very few Lie groups are relevant. In general, if we ask the homogeneous space to have additional structures the situation is of particular interest and it is natural to ask whether there is an impact on the corresponding Lie group. The converse might be even more interesting: is there a structure on a Lie group that acts on a manifold, that forces the manifold to be a homogeneous space? It turns out that we have to treat this on infinitesimal level, i.e. on the corresponding Lie algebra. These ideas are made concrete in Chapter 5, but shall suffice as a rough motivation right now.

The structures we mentioned are r -matrices. Those objects were developed in the course of integrable lattice systems. This is because r -matrix theory gives rise to Lax equations (see e.g. [18]). In order to motivate the notion of r -matrices we embed them in the theory of Lie bialgebras, which are themselves interesting for several reasons: first they occur as infinitesimal objects of Poisson-Lie groups that are involved again in integrable systems via monodromy matrices (consider for example [7]). We state the connection between Lie bialgebras and Poisson-Lie groups in Section 3.2 without proving this statement due to V. G. Drinfel'd. Second, Lie bialgebras are semi-classical limits of quantum groups and conversely any Lie bialgebra can be quantized in that way. This is the famous theorem by P. Etingof and D. Kazhdan (c.f. [38]). In the same manner R -matrices appear as quantizations of r -matrices (c.f. [39, Theorem 9.2]) and the quantum Yang-Baxter equation as a quantization of the classical Yang-Baxter equation (c.f. [39, Theorem 9.3]). We introduce the classical Yang-Baxter equation as a measure for r -matrices.

For Lie bialgebras we need some general Lie algebra cohomology and in particular the Chevalley-Eilenberg complex which we develop in Section 3.1 together with some well-known results in this topic. Then we are able to make a motivated definition and classification of Lie bialgebras in Section 3.2. Some interesting examples of Lie bialgebras finally lead to r -matrices in Section 3.3. We classify them and also consider the case when the corresponding Lie algebra is simple what will be helpful in the later sections. As a tool we define the Etingof-Schiffmann subalgebra in Section 3.4, a finite-dimensional Lie subalgebra in which the r -matrix is non-degenerate.

3.1 Lie Algebra Representations and Cohomology

Working with Lie algebras is interesting because they can act on other structures. We say that a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ over a field \mathbb{k} *acts* on a \mathbb{k} -vector space V if for every $x \in \mathfrak{g}$ there is a linear

map $\rho_x: V \rightarrow V$ such that the assignment $\rho: \mathfrak{g} \ni x \mapsto \rho_x \in \mathfrak{gl}(V)$ is a Lie algebra homomorphism, i.e. for all $x, y \in \mathfrak{g}$ one has

$$\rho_{[x,y]} = \rho_x \rho_y - \rho_y \rho_x, \quad (3.1.1)$$

where we endowed the general linear Lie algebra $\mathfrak{gl}(V)$ with the commutator. In this case we say that (ρ, V) is a *representation* of \mathfrak{g} or if we want to emphasize that the Lie algebra \mathfrak{g} acts on V we call V a \mathfrak{g} -*module*. We also use the short notation $x.v = \rho_x(v)$, for $x \in \mathfrak{g}$, $v \in V$. The following is inspired by [103].

Example 3.1.1 Let \mathfrak{g} be a Lie algebra over a field \mathbb{k} of characteristic zero. We give some examples that are of great interest in our topic.

- i.) First of all, \mathfrak{g} can act trivially on \mathbb{k} , i.e. $x.\lambda = 0 \in \mathbb{k}$ for all $x \in \mathfrak{g}$, $\lambda \in \mathbb{k}$. In this way \mathbb{k} becomes a \mathfrak{g} -module, called the *trivial module*.
- ii.) Furthermore, \mathfrak{g} can act on itself by the *adjoint action*, i.e. $\text{ad}_x(y) = [x, y]$ for $x, y \in \mathfrak{g}$. Indeed, \mathfrak{g} becomes a \mathfrak{g} -module in this way because the Lie bracket is bilinear by definition and Eq. (3.1.1) is just the Jacobi identity in this case.
- iii.) In the same fashion we can define adjoint actions of \mathfrak{g} on tensor powers of \mathfrak{g} . Explicitly, for $x \in \mathfrak{g}$ and factorizing tensors $y_1 \otimes \cdots \otimes y_n \in \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ we define

$$\text{ad}_x^{(n)}(y_1 \otimes \cdots \otimes y_n) = \sum_{k=1}^n y_1 \otimes \cdots \otimes \text{ad}_x(y_k) \otimes \cdots \otimes y_n \quad (3.1.2)$$

and extend this linearly to $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$. To prove Eq. (3.1.1) it is enough to consider factorizing elements $y_1 \otimes \cdots \otimes y_n \in \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$. By the Jacobi identity for \mathfrak{g} we get

$$\begin{aligned} & (\text{ad}_x^{(n)} \circ \text{ad}_y^{(n)} - \text{ad}_y^{(n)} \circ \text{ad}_x^{(n)})(y_1 \otimes \cdots \otimes y_n) \\ &= \text{ad}_x^{(n)} \left(\sum_{k=1}^n y_1 \otimes \cdots \otimes \text{ad}_y(y_k) \otimes \cdots \otimes y_n \right) \\ & - \text{ad}_y^{(n)} \left(\sum_{k=1}^n y_1 \otimes \cdots \otimes \text{ad}_x(y_k) \otimes \cdots \otimes y_n \right) \\ &= \sum_{k \neq j=1}^n y_1 \otimes \cdots \otimes \text{ad}_y(y_k) \otimes \cdots \otimes \text{ad}_x(y_j) \otimes \cdots \otimes y_n \\ & + \sum_{k=1}^n y_1 \otimes \cdots \otimes \text{ad}_x(\text{ad}_y(y_k)) \otimes \cdots \otimes y_n \\ & - \sum_{k \neq j=1}^n y_1 \otimes \cdots \otimes \text{ad}_x(y_k) \otimes \cdots \otimes \text{ad}_y(y_j) \otimes \cdots \otimes y_n \\ & - \sum_{k=1}^n y_1 \otimes \cdots \otimes \text{ad}_y(\text{ad}_x(y_k)) \otimes \cdots \otimes y_n \\ &= \sum_{k=1}^n (y_1 \otimes \cdots \otimes \text{ad}_x(\text{ad}_y(y_k)) \otimes \cdots \otimes y_n - y_1 \otimes \cdots \otimes \text{ad}_y(\text{ad}_x(y_k)) \otimes \cdots \otimes y_n) \\ &= \sum_{k=1}^n y_1 \otimes \cdots \otimes (\text{ad}_x(\text{ad}_y(y_k)) - \text{ad}_y(\text{ad}_x(y_k))) \otimes \cdots \otimes y_n \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n y_1 \otimes \cdots \otimes \operatorname{ad}_{[x,y]}(y_k) \otimes \cdots \otimes y_n \\
&= \operatorname{ad}_{[x,y]}^{(n)}(y_1 \otimes \cdots \otimes y_n),
\end{aligned}$$

where in the third equation the summand (k, j) of the first sum cancels with the summand (j, k) in the third sum. The linearity is obvious and for this $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ has the structure of a \mathfrak{g} -module. If not stated differently we always view $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ as this \mathfrak{g} -module (see Chapter 3.3). For example for $n = 2$ we get

$$\operatorname{ad}_x^{(2)}(y_1 \otimes y_2) = [x, y_1] \otimes y_2 + y_1 \otimes [x, y_2], \quad (3.1.3)$$

for $x \in \mathfrak{g}$, $y_1 \otimes y_2 \in \mathfrak{g} \otimes \mathfrak{g}$. This motivates the notations

$$\operatorname{ad}_x^{(2)}(v) = (\operatorname{ad}_x \otimes 1 + 1 \otimes \operatorname{ad}_x)(v) = [x \otimes 1 + 1 \otimes x, v], \quad (3.1.4)$$

for $x \in \mathfrak{g}$, $v \in \mathfrak{g} \otimes \mathfrak{g}$, where $1: \mathfrak{g} \rightarrow \mathfrak{g}$ is the identity map.

Having in mind these examples we consider again some arbitrary \mathfrak{g} -module V , where \mathfrak{g} is a Lie algebra over a field \mathbb{k} of characteristic zero. For a natural number $n \in \mathbb{N}$ we denote by $C^n(\mathfrak{g}, V) = \operatorname{Hom}_{\mathbb{k}}(\Lambda^n \mathfrak{g}, V)$ the vector space of \mathbb{k} -linear maps from the n -th exterior power $\Lambda^n \mathfrak{g}$ of \mathfrak{g} to V . The elements of $C^n(\mathfrak{g}, V)$ are called *n -cochains on \mathfrak{g} with values in V* (or just *n -cochains* if there is no confusion which Lie algebra module is meant). We define a mapping $\delta_n: C^n(\mathfrak{g}, V) \rightarrow C^{n+1}(\mathfrak{g}, V)$ from the set of n -cochains to the set of $(n+1)$ -cochains by

$$\begin{aligned}
(\delta_n c)(x_1 \wedge \cdots \wedge x_{n+1}) &= \sum_{k=1}^{n+1} x_k \cdot c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{n+1}) \\
&\quad + \sum_{k < j} c([x_k, x_j] \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{n+1}),
\end{aligned}$$

for all $c \in C^n(\mathfrak{g}, V)$ and $x_1 \wedge \cdots \wedge x_{n+1} \in \Lambda^n \mathfrak{g}$. The point in the first sum stands of course for the representation ρ corresponding to the \mathfrak{g} -module structure of V , i.e. $x_k \cdot c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{n+1}) = \rho_{x_k}(c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{n+1}))$ and \hat{x}_k denotes that the element x_k is removed in the exterior product. In particular, we are interested in the kernel of δ_n . We denote this vector space by $Z_n(\mathfrak{g}, V) = \ker(\delta_n)$ and call its elements *n -cocycles on \mathfrak{g} with values in V* or just *n -cocycles*. This space can be compared to the image of δ_{n-1} , i.e. the vector space $B_n(\mathfrak{g}, V) = \operatorname{im}(\delta_{n-1})$ whose elements are said to be the *n -coboundaries on \mathfrak{g} with values in V* or again just *n -coboundaries*, where we set $B_0(\mathfrak{g}, V) = \{0\}$. The reader can check that $\delta_{n+1} \circ \delta_n = 0$. We often just write $\delta = \delta_n$. The equation $\delta^2 = 0$ implies $B_n(\mathfrak{g}, V) \subseteq Z_n(\mathfrak{g}, V) \subseteq C^n(\mathfrak{g}, V)$, i.e. every n -coboundary is a n -cocycle. Then the set

$$H^n(\mathfrak{g}, V) = Z_n(\mathfrak{g}, V) / B_n(\mathfrak{g}, V) \quad (3.1.5)$$

is a well-defined \mathbb{k} -vector space.

Definition 3.1.2 *Let (ρ, V) be a Lie algebra representation of \mathfrak{g} . The tuple $(C^n(\mathfrak{g}, V), \delta_n)_{n \in \mathbb{N}_0}$ is said to be the **Chevalley-Eilenberg complex** and $\delta_n = \delta$ the **Chevalley-Eilenberg differential**. Moreover, one calls $H^n(\mathfrak{g}, V)$ the **n -th cohomology group on \mathfrak{g} with values in V** , or just the **n -th cohomology group**.*

Remark 3.1.3 Let us consider the low order cohomologies.

i.) For $n = 0$ one has $c \in H^0(\mathfrak{g}, V) = Z_0(\mathfrak{g}, V) \subseteq V$ if and only if

$$0 = \delta c(x) = x.c,$$

for all $x \in \mathfrak{g}$. This vector space is sometimes denoted by $V^{\mathfrak{g}}$. Thus the 0-th cohomology group consists of the elements of V that are annihilated by all elements of \mathfrak{g} via the corresponding \mathfrak{g} -module representation, i.e.

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}} \subseteq V. \quad (3.1.6)$$

If \mathfrak{g} acts trivially on \mathbb{k} we obtain $H^0(\mathfrak{g}, \mathbb{k}) = \mathbb{k}$. If \mathfrak{g} acts on \mathfrak{g} by the adjoint action we get

$$H^0(\mathfrak{g}, \mathfrak{g}) = \{y \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } x \in \mathfrak{g}\} \quad (3.1.7)$$

and if \mathfrak{g} acts on $\mathfrak{g} \otimes \mathfrak{g}$ by $\text{ad}^{(2)}$ we see that

$$H^0(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) = \text{span}_{\mathbb{k}}\{y_1 \otimes y_2 \in \mathfrak{g} \otimes \mathfrak{g} \mid [y_1, x] \otimes y_2 = y_1 \otimes [x, y_2] \text{ for all } x \in \mathfrak{g}\}. \quad (3.1.8)$$

ii.) For $n = 1$ we observe that $c \in Z_1(\mathfrak{g}, V)$ if and only if for all $x_1, x_2 \in \mathfrak{g}$

$$0 = (-1)^{1+1}x_1.c(x_2) + (-1)^{2+1}x_2.c(x_1) + (-1)^{1+2}c([x_1, x_2]) \quad (3.1.9)$$

or equivalently

$$c([x_1, x_2]) = x_1.c(x_2) - x_2.c(x_1), \quad (3.1.10)$$

i.e. if and only if $c: \mathfrak{g} \rightarrow V$ is a *derivation*. We denote the vector space of derivations on \mathfrak{g} with values in V by $\text{Der}(\mathfrak{g}, V)$. Then $Z_1(\mathfrak{g}, V) = \text{Der}(\mathfrak{g}, V)$. Moreover, $c \in B_1(\mathfrak{g}, V)$ if and only if there is an element $b \in C^0(\mathfrak{g}, V) = V$ such that $c(x) = \delta b(x) = x.b$ for all $x \in \mathfrak{g}$. Such elements are said to be the *inner derivations* on \mathfrak{g} with values in V and the vector space of them is denoted by $\text{IDer}(\mathfrak{g}, V)$. Thus $B_1(\mathfrak{g}, V) = \text{IDer}(\mathfrak{g}, V)$ and

$$H^1(\mathfrak{g}, V) = \text{Der}(\mathfrak{g}, V) / \text{IDer}(\mathfrak{g}, V) = \text{Out}(\mathfrak{g}, V), \quad (3.1.11)$$

where the quotient of derivations and inner derivations results in the so-called *outer derivations* $\text{Out}(\mathfrak{g}, V)$ on \mathfrak{g} with values in V . In particular, if \mathfrak{g} acts trivially on \mathbb{k} we obtain

$$H^1(\mathfrak{g}, \mathbb{k}) = \{c \in C^1(\mathfrak{g}, \mathbb{k}) \mid c([x_1, x_2]) = 0 \text{ for all } x_1, x_2 \in \mathfrak{g}\} / \{0\} = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*, \quad (3.1.12)$$

where the star denotes the dual space and $[\mathfrak{g}, \mathfrak{g}]$ the \mathbb{k} -span of all brackets $[x, y]$, for some $x, y \in \mathfrak{g}$.

iii.) For $n = 2$ one can prove (c.f [78, Proposition I.3.]) that

$$H^2(\mathfrak{g}, V) \cong \text{Ext}(\mathfrak{g}, V), \quad (3.1.13)$$

where $\text{Ext}(\mathfrak{g}, V)$ is the set of equivalence classes of abelian extensions of \mathfrak{g} by V . If \mathfrak{g} is a simple Lie algebra over \mathbb{C} and $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ the *Casimir element* associated to a non-degenerate invariant form on \mathfrak{g} we get

$$(\Lambda^3 \mathfrak{g})^{\mathfrak{g}} = \mathbb{C}Z, \quad (3.1.14)$$

where $Z = [\Omega \otimes 1, 1 \otimes \Omega]$.

Many details about Lie algebra cohomology can be found in [51], [78] and [84]. To end this section we want to state some famous results about Lie algebra cohomology.

Theorem 3.1.4 *Let \mathfrak{g} be a complex semisimple Lie algebra and V a finite-dimensional \mathfrak{g} -module. Then $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = \{0\}$. If $V^{\mathfrak{g}} = \{0\}$ we even get $H^n(\mathfrak{g}, V) = \{0\}$ for all $n \in \mathbb{N}_0$.*

PROOF: The first statement is known as **Whitehead's Lemma** and can be found in [102]. The second statement is a consequence of this lemma (c.f. [52]). \square

Remark 3.1.5 In [24] it is proven that if \mathfrak{g} is a nilpotent Lie algebra of dimension $n \in \mathbb{N}$ the dimension of $H^k(\mathfrak{g}, \mathbb{k})$ is at least 2 for $1 \leq k \leq n-1$. This gives a helpful intuition that nilpotent Lie algebras have “big” Lie algebra cohomologies while the Lie algebra cohomologies of semisimple Lie algebras are “small”. Finally, for a semisimple Lie algebras there is a splitting of $H^n(\mathfrak{g}, V)$ into 1-dimensional modules with respect to a root decomposition of \mathfrak{g} . This is discussed in [64].

3.2 Lie Bialgebras and Poisson-Lie Groups

Lie bialgebras and Poisson-Lie groups are well-studied objects. We refer to the first section of the notes [62] of Y. Kosmann-Schwarzbach for a introduction to Lie bialgebras and to the fourth section to examine Poisson-Lie groups. Further sources are [27, Chapter 1], [39, Chapter 2], [67, Chapter 11], [70, Chapter 8] and [101, Chapter 10].

Consider a Lie algebra \mathfrak{g} over a field \mathbb{k} of characteristic zero. We want to equip \mathfrak{g} with some additional structure. As motivated from the last section we regard a linear map

$$\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \quad (3.2.1)$$

that is a 1-cocycle on \mathfrak{g} with values in $\mathfrak{g} \otimes \mathfrak{g}$, where we view $\mathfrak{g} \otimes \mathfrak{g}$ as a \mathfrak{g} -module via $\text{ad}^{(2)}$ as usual, i.e. $\gamma \in Z_1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$. It is common to use the short notation

$$\gamma(x) = \sum x_{[1]} \otimes x_{[2]} = x_{[1]} \otimes x_{[2]} \quad (3.2.2)$$

for $x \in \mathfrak{g}$ to represent the element $\gamma(x) \in \mathfrak{g} \otimes \mathfrak{g}$. The *cocycle condition* reads

$$\delta\gamma = 0 \quad (3.2.3)$$

or equivalently

$$\gamma([x, y]) = \text{ad}_x^{(2)}(\gamma(y)) - \text{ad}_y^{(2)}(\gamma(x)), \quad (3.2.4)$$

for all $x, y \in \mathfrak{g}$. If we require γ to satisfy in addition

$$x_{[1]} \otimes x_{[2]} = -x_{[2]} \otimes x_{[1]} \quad (3.2.5)$$

and

$$\text{Alt}((\gamma \otimes 1)\gamma(x)) = 0 \quad (3.2.6)$$

for all $x \in \mathfrak{g}$, where Alt is the \mathbb{k} -linear function $\bigotimes^3 \mathfrak{g} \rightarrow \bigotimes^3 \mathfrak{g}$ that satisfies on factorizing tensors $\text{Alt}(x \otimes y \otimes z) = x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y$, the Lie algebra \mathfrak{g} becomes also a Lie coalgebra, or equivalently the dual \mathfrak{g}^* of \mathfrak{g} gets a Lie algebra. The Lie bracket on \mathfrak{g}^* is given by the left transpose of γ , i.e.

$${}^t\gamma: \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^* \quad (3.2.7)$$

is a Lie bracket on \mathfrak{g}^* . Indeed we can define a bracket $[\cdot, \cdot]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ on \mathfrak{g}^* by

$$[\xi, \eta]_{\mathfrak{g}^*} = {}^t\gamma(\xi \otimes \eta), \quad (3.2.8)$$

for all $\xi, \eta \in \mathfrak{g}^*$. If we introduce the non-degenerate dual pairing $\langle \cdot, \cdot \rangle : \bigotimes^n \mathfrak{g}^* \times \bigotimes^n \mathfrak{g} \rightarrow \mathbb{k}$ we can prove that Eq. (3.2.8) defines a Lie bracket on \mathfrak{g}^* if and only if γ satisfies conditions (3.2.5) and (3.2.6). Of course condition (3.2.5) can immediately be absorbed if we require γ to be a map $\mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$. The skew-symmetry of $[\cdot, \cdot]_{\mathfrak{g}^*}$ follows from condition (3.2.5) because for all $x \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$ one has

$$\begin{aligned} \langle [\xi, \eta]_{\mathfrak{g}^*}, x \rangle &= \langle {}^t\gamma(\xi \otimes \eta), x \rangle = \langle \xi \otimes \eta, \gamma(x) \rangle \\ &= \langle \xi \otimes \eta, x_{[1]} \otimes x_{[2]} \rangle = \langle \eta \otimes \xi, x_{[2]} \otimes x_{[1]} \rangle \\ &= -\langle \eta \otimes \xi, x_{[1]} \otimes x_{[2]} \rangle = -\langle {}^t\gamma(\eta, \xi), x \rangle \\ &= \langle -[\eta, \xi]_{\mathfrak{g}^*}, x \rangle. \end{aligned}$$

This implies $[\xi, \eta]_{\mathfrak{g}^*} = -[\eta, \xi]_{\mathfrak{g}^*}$ because the dual pairing is non-degenerate. Arranging the above equations in another order we see that the skew-symmetry of $[\cdot, \cdot]_{\mathfrak{g}^*}$ implies condition (3.2.5). Moreover, the Jacobi identity of $[\cdot, \cdot]_{\mathfrak{g}^*}$ is equivalent to condition (3.2.6), the so called *coJacobi identity*. To prove this, observe that for $x \in \mathfrak{g}$ and $\xi, \eta, \zeta \in \mathfrak{g}^*$ one has

$$\begin{aligned} \langle [[\xi, \eta]_{\mathfrak{g}^*}, \zeta]_{\mathfrak{g}^*}, x \rangle &= \langle {}^t\gamma({}^t\gamma(\xi \otimes \eta) \otimes \zeta), x \rangle = \langle {}^t\gamma(\xi \otimes \eta) \otimes \zeta, \gamma(x) \rangle \\ &= \langle \xi \otimes \eta \otimes \zeta, (\gamma \otimes 1) \circ \gamma(x) \rangle. \end{aligned}$$

Thus every term of the coJacobi identity corresponds to one term of the Jacobi identity of $[\cdot, \cdot]_{\mathfrak{g}^*}$ and the equivalence is proved. After recognizing the powerful impact of the above considerations we give the well-motivated definition of a Lie bialgebra.

Definition 3.2.1 (Lie Bialgebra) A Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ is a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ together with a map

$$\gamma : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g} \quad (3.2.9)$$

satisfying the coJacobi identity (3.2.6) and the cocycle identity (3.2.4). The cocycle γ is called the Lie bialgebra structure of $(\mathfrak{g}, [\cdot, \cdot], \gamma)$.

We already proved the following proposition in the above lines.

Proposition 3.2.2 A Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ endowed with a 1-cocycle $\gamma : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ if and only if

$$[\xi, \eta]_{\mathfrak{g}^*} = {}^t\gamma(\xi \otimes \eta), \quad (3.2.10)$$

where $\xi, \eta \in \mathfrak{g}^*$, defines a Lie bracket on \mathfrak{g}^* .

Every new object goes hand in hand with its morphisms and substructures.

Definition 3.2.3 Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \gamma_{\mathfrak{g}})$ and $(\mathfrak{k}, [\cdot, \cdot]_{\mathfrak{k}}, \gamma_{\mathfrak{k}})$ be two Lie bialgebras.

i.) A Lie algebra morphism $\phi : \mathfrak{g} \rightarrow \mathfrak{k}$ is said to be a **morphism of Lie bialgebras** if it respects the cocycles, i.e.

$$(\phi \otimes \phi)(\gamma_{\mathfrak{g}}(x)) = \gamma_{\mathfrak{k}}(\phi(x)), \quad (3.2.11)$$

for all $x \in \mathfrak{g}$. The condition being a Lie algebra morphism reads

$$\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{k}}, \quad (3.2.12)$$

for all $x, y \in \mathfrak{g}$.

ii.) A Lie subalgebra \mathfrak{h} of \mathfrak{g} is said to be a **Lie subbialgebra** if the cocycle on \mathfrak{g} restricts to a map

$$\gamma_{\mathfrak{g}}: \mathfrak{h} \rightarrow \mathfrak{h} \wedge \mathfrak{h}, \quad (3.2.13)$$

i.e. $\gamma_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{h} \wedge \mathfrak{h}$. The condition being a Lie subalgebra of \mathfrak{g} reads $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{g}} \subseteq \mathfrak{h}$.

iii.) A Lie ideal \mathfrak{h} of \mathfrak{g} is said to be **Lie coideal** if

$$\gamma_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g} \otimes \mathfrak{h} + \mathfrak{h} \otimes \mathfrak{g}. \quad (3.2.14)$$

The condition being a Lie ideal of \mathfrak{g} is that \mathfrak{h} is a left ideal of \mathfrak{g} with respect to the Lie bracket (or equivalently a right ideal or an ideal), i.e. $[\mathfrak{h}, \mathfrak{g}]_{\mathfrak{g}} \subseteq \mathfrak{h}$.

Remark that if \mathfrak{h} is a Lie ideal of $(\mathfrak{g}, [\cdot, \cdot])$ the quotient Lie algebra $(\mathfrak{g}/\mathfrak{h}, [\cdot, \cdot])$ inherits a Lie bialgebra structure from \mathfrak{g} if and only if \mathfrak{h} is a Lie coideal. We already realized in Proposition 3.2.2 that the cocycle $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ on \mathfrak{g} that satisfies the coJacobi identity is equivalent to a Lie bracket on the dual \mathfrak{g}^* of \mathfrak{g} . We expand this result in the sense that if γ satisfies the above conditions and the cocycle condition, not only \mathfrak{g} but also \mathfrak{g}^* has the structure of a Lie bialgebra. Thus the notion of Lie bialgebra is totally self dual.

Proposition 3.2.4 For a finite-dimensional Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ consider the dual maps

$${}^t[\cdot, \cdot]: \mathfrak{g}^* \rightarrow \mathfrak{g}^* \wedge \mathfrak{g}^* \text{ and } {}^t\gamma: \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^* \quad (3.2.15)$$

of $[\cdot, \cdot]$ and γ . Then $(\mathfrak{g}^*, {}^t\gamma, {}^t[\cdot, \cdot])$ is a Lie bialgebra, where ${}^t\gamma$ is the Lie bracket and ${}^t[\cdot, \cdot]$ the cocycle on \mathfrak{g}^* .

PROOF: We already proved that $(\mathfrak{g}^*, {}^t\gamma)$ is a Lie algebra if and only if γ satisfies conditions (3.2.5) and (3.2.6). The only thing left to prove the proposition is that ${}^t[\cdot, \cdot]$ is a cocycle. The restriction to finite-dimensional Lie bialgebras is necessary if we want ${}^t({}^t[\cdot, \cdot])$ to define the Lie bracket $[\cdot, \cdot]$ on $(\mathfrak{g}^*)^* = \mathfrak{g}$ again. Let us compute for $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$

$$\begin{aligned} \langle {}^t[\cdot, \cdot]({}^t\gamma(\xi \wedge \eta)), x \wedge y \rangle &= \langle {}^t\gamma(\xi \wedge \eta), [x, y] \rangle \\ &= \langle \xi \wedge \eta, \gamma([x, y]) \rangle \\ &= \langle \xi \wedge \eta, (\text{ad}_x^{(2)}(\gamma(y)) - \text{ad}_y^{(2)}(\gamma(x))) \rangle \\ &= \langle \xi \wedge \eta, ([x, y_{[1]}] \wedge y_{[2]} + y_{[1]} \wedge [x, y_{[2]}] \\ &\quad - [y, x_{[1]}] \wedge x_{[2]} - x_{[1]} \wedge [y, x_{[2]}]) \rangle \\ &= \langle ({}^t[\cdot, \cdot](\xi) \wedge \eta - \xi \wedge ({}^t[\cdot, \cdot](\eta))), x \wedge y_{[1]} \wedge y_{[2]} \rangle \\ &\quad - \langle ({}^t[\cdot, \cdot](\xi) \wedge \eta - \xi \wedge ({}^t[\cdot, \cdot](\eta))), x_{[1]} \wedge x_{[2]} \wedge y \rangle \\ &= \langle (-\xi_{[1]} \wedge \eta \wedge \xi_{[2]} + \eta_{[1]} \wedge \xi \wedge \eta_{[2]}), x \wedge \gamma(y) \rangle \\ &\quad - \langle (\eta \wedge \xi_{[1]} \wedge \xi_{[2]} - \xi \wedge \eta_{[1]} \wedge \eta_{[2]}), \gamma(x) \wedge y \rangle \\ &= \langle ([\xi, \eta_{[1]}]_{\mathfrak{g}^*} \wedge \eta_{[2]} + \eta_{[1]} \wedge [\xi, \eta_{[2]}]_{\mathfrak{g}^*} \\ &\quad - ([\eta, \xi_{[1]}]_{\mathfrak{g}^*} \wedge \xi_{[2]} + \xi_{[1]} \wedge [\eta, \xi_{[2]}]_{\mathfrak{g}^*})), x \wedge y \rangle \\ &= \langle ((\text{ad}_{\xi}^*)^{(2)}({}^t[\cdot, \cdot](\eta)) - (\text{ad}_{\eta}^*)^{(2)}({}^t[\cdot, \cdot](\xi))), x \wedge y \rangle. \end{aligned}$$

The claim follows because the dual pairing is non-degenerate. Then ${}^t[\cdot, \cdot]$ is a cocycle on \mathfrak{g}^* if γ is a cocycle on \mathfrak{g} . Using the equations above in different order one can check that if ${}^t[\cdot, \cdot]$ is a cocycle on \mathfrak{g}^* also γ is a cocycle on \mathfrak{g} and in the finite-dimensional case these are indeed equivalent conditions. A nice pictorial proof of this statement can be found in [39, Proposition 2.2]. \square

Now we can introduce a first example of a Lie bialgebra.

Example 3.2.5 Consider a noncommutative Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ of dimension 2 over a field \mathbb{k} of characteristic zero and take a basis $\{X, Y\}$ of \mathfrak{g} such that

$$[X, Y] = X. \quad (3.2.16)$$

By the bilinearity and antisymmetry of $[\cdot, \cdot]$ condition (3.2.16) determines the Lie bracket completely. Define a map $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ by $\gamma(X) = 0$, $\gamma(Y) = -X \wedge Y$ and linear expansion and prove that this structures $(\mathfrak{g}, [\cdot, \cdot])$ as a Lie bialgebra. Of course it is enough to prove this just for the basis elements. Indeed, the cocycle condition holds:

$$\begin{aligned} \text{ad}_X^{(2)}(\gamma(Y)) - \text{ad}_Y^{(2)}(\gamma(X)) &= \text{ad}_X^{(2)}(-X \wedge Y) - 0 \\ &= \text{ad}_X^{(2)}(Y \otimes X) - \text{ad}_X^{(2)}(X \otimes Y) \\ &= [X, Y] \otimes X + Y \otimes [X, X] - [X, X] \otimes Y - X \otimes [X, Y] \\ &= X \otimes X - X \otimes X \\ &= 0 \\ &= \gamma(X) \\ &= \gamma([X, Y]). \end{aligned}$$

We also check the coJacobi identity for X

$$\text{Alt}((\gamma \otimes 1)\gamma(X)) = \text{Alt}(0) = 0$$

and for Y

$$\begin{aligned} \text{Alt}((\gamma \otimes 1)\gamma(Y)) &= \text{Alt}((\gamma \otimes 1)(Y \otimes X - X \otimes Y)) \\ &= \text{Alt}((Y \otimes X - X \otimes Y) \otimes X) - 0 \\ &= Y \otimes X \otimes X + X \otimes Y \otimes X + X \otimes X \otimes Y \\ &\quad - X \otimes Y \otimes X - X \otimes X \otimes Y - Y \otimes X \otimes X \\ &= 0. \end{aligned}$$

Thus $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ is a Lie bialgebra.

There are many more examples of Lie bialgebras. Some of them can be found in [39, Section 2.2.2], [62, Section 1] or [70, Section 8]. A very important motivation for Lie bialgebras comes from the famous theorem of Sophus Lie, stating that the categories of simply connected Lie groups and finite-dimensional Lie algebras are equivalent. This gives the *Lie functor*

$$\text{Lie}(G) = \mathfrak{g}, \quad (3.2.17)$$

that assigns to every simply connected Lie group G its infinitesimal counterpart, i.e. a finite-dimensional Lie algebra \mathfrak{g} . Our question is now: what is the global counterpart of a Lie bialgebra? To answer this we give a very short repetition on Poisson manifolds. For a conceptual introduction consider for example [67], [101] or [105]. A smooth manifold M together with a *Poisson bracket*

$$\{\cdot, \cdot\}: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \quad (3.2.18)$$

for its smooth real-valued functions $\mathcal{C}^\infty(M)$ is said to be a *Poisson manifold*. The axioms that the \mathbb{k} -bilinear map in Eq. (3.2.18) has to satisfy to define a Poisson bracket are skew-symmetry, Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\} \quad (3.2.19)$$

and the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad (3.2.20)$$

for all $f, g, h \in \mathcal{C}^\infty(M)$. The smooth functions on M are viewed as a *Poisson algebra* via its commutative associative algebra structure of pointwise multiplication together with a Poisson bracket. Such Poisson brackets are in one-to-one correspondence with bivector fields $\pi \in \Gamma^\infty(\Lambda^2 TM)$ on M via

$$\{f, g\} = \pi(df, dg) \quad (3.2.21)$$

for all $f, g \in \mathcal{C}^\infty(M)$, where d denotes the exterior derivative. For a Poisson manifold $(M, \{\cdot, \cdot\})$ we call π the corresponding *Poisson bivector* or *Poisson structure*. A smooth map $\Phi: (M_1, \{\cdot, \cdot\}_1) \rightarrow (M_2, \{\cdot, \cdot\}_2)$ between two Poisson manifolds $(M_1, \{\cdot, \cdot\}_1)$ and $(M_2, \{\cdot, \cdot\}_2)$ is said to be a *Poisson map* if for all $f, g \in \mathcal{C}^\infty(M_2)$ one has

$$\Phi^*\{f, g\}_2 = \{\Phi^*f, \Phi^*g\}_1 \quad (3.2.22)$$

where Φ^* denotes the *pull back* of Φ , i.e. $\Phi^*f = f \circ \Phi$ for all $f \in \mathcal{C}^\infty(M_2)$. A very important class of examples of Poisson manifolds is given by symplectic manifolds. Remember that a *symplectic manifold* is a smooth manifold M together with a closed non-degenerated 2-form $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$, called the *symplectic structure*. In fact, for any symplectic structure $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$ we can always define a Poisson bracket by

$$\{f, g\} = \omega(X_f, X_g) \quad (3.2.23)$$

for all $f, g \in \mathcal{C}^\infty(M)$. The objects X_f and X_g in Eq. (3.2.23) are said to be the *Hamiltonian vector fields* of f and g , respectively and they are defined by

$$X_f = (df)^\sharp, \quad (3.2.24)$$

where $\sharp: T^*M \rightarrow TM$ is the *musical isomorphism*. Thus another way of writing Eq. (3.2.23) is

$$\{f, g\} = X_g f, \quad (3.2.25)$$

for $f, g \in \mathcal{C}^\infty(M)$. It is known that for any $p \in M$ the map

$$\tilde{\pi}: T_p^*M \ni \alpha_p \mapsto \pi_p(\cdot, \alpha_p) \in T_pM \quad (3.2.26)$$

is surjective if (M, ω) is a symplectic manifold and that

$$\pi(df, dg) = \omega(X_f, X_g) \quad (3.2.27)$$

holds for all $f, g \in \mathcal{C}^\infty(M)$.

An interesting structure occurs if we combine the concept of Poisson manifolds with the notion of Lie groups.

Definition 3.2.6 (Poisson-Lie Group) A Poisson manifold $(G, \{\cdot, \cdot\})$ that is also a Lie group is said to be a Poisson-Lie group if the group multiplication $m: G \times G \rightarrow G$ is a Poisson map.

Remark 3.2.7 Let $(G, \{\cdot, \cdot\})$ be a Poisson-Lie group and denote by $\lambda_u: G \ni v \mapsto uv \in G$ and $\rho_u: G \ni v \mapsto vu \in G$ the left and right multiplication on G with some element $u \in G$, respectively. Then the group multiplication m is a Poisson map, i.e.

$$\{f, g\}(uv) = \{f, g\}(m(u \times v)) = \{f \circ \lambda_u, g \circ \lambda_u\}(v) + \{f \circ \rho_v, g \circ \rho_v\}(u), \quad (3.2.28)$$

where $f, g \in \mathcal{C}^\infty(G)$ and $u, v \in G$. There is also an equivalent condition to (3.2.28) for the corresponding Poisson bivector $\pi \in \Gamma^\infty(\Lambda^2 TG)$, namely

$$\pi(uv) = (T_u \rho_v \otimes T_u \rho_v) \pi(u) + (T_v \lambda_u \otimes T_v \lambda_u) \pi(v), \quad (3.2.29)$$

for all $u, v \in G$. In particular, we see that if G is a Poisson-Lie group with identity element $e \in G$ we get

$$\pi(e) = 0. \quad (3.2.30)$$

After this recap we come back to our question: a result by Drinfel'd shows the connection between Poisson-Lie groups and Lie bialgebras in analogy to the Lie functor given in Eq. (3.2.17).

Theorem 3.2.8 (Drinfel'd) There is a functor

$$\text{Drin}: G \mapsto \text{Lie}(G) \quad (3.2.31)$$

between the categories of simply connected Poisson-Lie groups and finite-dimensional Lie bialgebras and this functor gives an equivalence of these categories.

Thus, one can see Lie bialgebras in a bigger context. The proof is not necessary for our purpose and we just refer to [39, Theorem 2.2]. For instance, this theorem motivates the notion of the dual G^* of a Poisson-Lie group G , where $\mathfrak{g}^* = \text{Drin}(G^*)$ is the dual of $\mathfrak{g} = \text{Drin}(G)$. In the next section we see some special kinds of Lie bialgebras. They have global counterparts in conformity with the functor of the last theorem. Consider [39, Section 9.4] for the definitions of these so called *coboundary Hopf algebras* and the connection to their infinitesimal objects.

3.3 Coboundary Lie Bialgebras and the Classical Yang-Baxter Equation

Here we give a nice connection of the last two sections. For convenient literature we refer to [27, Chapter 2], [39, Chapter 3] and [62, Section 2]. Consider Lie bialgebras $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ with a cocycle γ on \mathfrak{g} with values in $\mathfrak{g} \otimes \mathfrak{g}$. How does the situation change when γ is also a coboundary on \mathfrak{g} with values in $\mathfrak{g} \otimes \mathfrak{g}$, i.e. if there is an element $r \in C^0(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}$ such that $\gamma = \delta r$? And what conditions has an arbitrary element $r \in \mathfrak{g} \otimes \mathfrak{g}$ to obey to define a cocycle $\gamma = \delta r$ such that $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ becomes a Lie bialgebra? Because $\delta^2 = 0$ the element δr will automatically be a cocycle, but conditions (3.2.5) and (3.2.6) do not have to be valid in the general case. A powerful instrument to measure if the coJacobi identity is also given is the so called *classical Yang-Baxter map* CYB: $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^{\otimes 3}$ defined by

$$\text{CYB}(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \quad (3.3.1)$$

for all elements $r = \sum r_1 \otimes r_2 = r_1 \otimes r_2 \in \mathfrak{g} \otimes \mathfrak{g}$ (we use a notation very similar to (3.2.2)), where we defined

$$\begin{aligned} r_{12} &= r_1 \otimes r_2 \otimes 1, \\ r_{13} &= r_1 \otimes 1 \otimes r_2, \\ r_{23} &= 1 \otimes r_1 \otimes r_2 \in \mathcal{U}(\mathfrak{g})^{\otimes 3} \end{aligned}$$

on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} . The brackets in (3.3.1) are intuitive abbreviations defined by

$$\begin{aligned} [r_{12}, r_{13}] &= [r_1, r'_1] \otimes r_2 \otimes r'_2, \\ [r_{12}, r_{23}] &= r_1 \otimes [r_2, r'_1] \otimes r'_2, \\ [r_{13}, r_{23}] &= r_1 \otimes r'_1 \otimes [r_2, r'_2], \end{aligned}$$

where we denoted the second summation by $r = r'_1 \otimes r'_2$. With this map we are able to determine whenever r gives rise to a Lie bialgebra via the above construction.

Theorem 3.3.1 *Given a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ that is skew-symmetric. The triple $(\mathfrak{g}, [\cdot, \cdot], \delta r)$ is a Lie bialgebra if and only if $\text{CYB}(r) \in \mathfrak{g}^{\otimes 3}$ is a \mathfrak{g} -invariant element. The condition for $\text{CYB}(r)$ to be \mathfrak{g} -invariant reads*

$$\text{ad}_x^{(3)}(\text{CYB}(r)) = [x, \text{CYB}(r)] = 0, \quad (3.3.2)$$

for all $x \in \mathfrak{g}$.

PROOF: These lines are inspired by [62, Proposition on page 17]. A geometric proof can be found in [39, Theorem 3.1]. Our aim is to prove for any $x \in \mathfrak{g}$ the identity

$$\text{Alt}((\gamma \otimes 1)\gamma(x)) = -\text{ad}_x^{(3)}(\text{CYB}(r)) \quad (3.3.3)$$

by only using the skew-symmetry and Jacobi identity of $[\cdot, \cdot]$, the skew-symmetry of r and

$$\gamma(x) = \delta r(x) = \text{ad}_x^{(2)}r. \quad (3.3.4)$$

We start writing out the left side of Eq. (3.3.3):

$$\gamma(x) = \text{ad}_x^{(2)}r = [x, r_1] \otimes r_2 + r_1 \otimes [x, r_2]$$

implies

$$\begin{aligned} (\gamma \otimes 1)\gamma(x) &= \gamma([x, r_1]) \otimes r_2 + \gamma(r_1) \otimes [x, r_2] \\ &= \text{ad}_{[x, r_1]}^{(2)}r \otimes r_2 + \text{ad}_{r_1}^{(2)}r \otimes [x, r_2] \\ &= [[x, r_1], r'_1] \otimes r'_2 \otimes r_2 + r'_1 \otimes [[x, r_1], r'_2] \otimes r_2 \\ &\quad + [r_1, r'_1] \otimes r'_2 \otimes [x, r_2] + r'_1 \otimes [r_1, r'_2] \otimes [x, r_2]. \end{aligned}$$

To separate the two summations over the tensor components of r we denoted the second one by $r'_1 \otimes r'_2$. After applying Alt we obtain twelve terms. The skew-symmetry of $[\cdot, \cdot]$ and r imply

$$\begin{aligned} \text{Alt}((\gamma \otimes 1)\gamma(x)) &= [[x, r_1], r'_1] \otimes r'_2 \otimes r_2 - r_1 \otimes [[x, r_2], r'_1] \otimes r'_2 + r'_1 \otimes r_1 \otimes [[x, r_2], r'_2] \\ &\quad + r'_1 \otimes [[x, r_1], r'_2] \otimes r_2 - r_1 \otimes r'_1 \otimes [[x, r_2], r'_2] - [[x, r_1], r'_1] \otimes r_2 \otimes r'_2 \end{aligned}$$

$$\begin{aligned}
& - [r'_1, r_1] \otimes r'_2 \otimes [x, r_2] + [x, r_2] \otimes [r_1, r'_1] \otimes r'_2 + r'_1 \otimes [x, r_2] \otimes [r'_2, r_1] \\
& - r'_1 \otimes [r'_2, r_1] \otimes [x, r_2] - [x, r_1] \otimes r'_1 \otimes [r_2, r'_2] - [r_1, r'_1] \otimes [x, r_2] \otimes r'_2.
\end{aligned}$$

We know that

$$\text{CYB}(r) = [r_1, r'_1] \otimes r_2 \otimes r'_2 + r_1 \otimes [r_2, r'_1] \otimes r'_2 + r_1 \otimes r'_1 \otimes [r_2, r'_2],$$

so the right hand side of Eq. (3.3.3) reads

$$\begin{aligned}
\text{ad}_x^{(3)}(\text{CYB}(r)) &= [x, [r_1, r'_1]] \otimes r_2 \otimes r'_2 + [r_1, r'_1] \otimes [x, r_2] \otimes r'_2 + [r_1, r'_1] \otimes r_2 \otimes [x, r'_2] \\
&+ [x, r_1] \otimes [r_2, r'_1] \otimes r'_2 + r_1 \otimes [x, [r_2, r'_1]] \otimes r'_2 + r_1 \otimes [r_2, r'_1] \otimes [x, r'_2] \\
&+ [x, r_1] \otimes r'_1 \otimes [r_2, r'_2] + r_1 \otimes [x, r'_1] \otimes [r_2, r'_2] + r_1 \otimes r'_1 \otimes [x, [r_2, r'_2]].
\end{aligned}$$

We apply the Jacobi identity to the terms with double Lie bracket and get

$$\begin{aligned}
\text{ad}_x^{(3)}(\text{CYB}(r)) &= [[x, r_1], r'_1] \otimes r_2 \otimes r'_2 - [[x, r'_1], r_1] \otimes r_2 \otimes r'_2 \\
&+ r_1 \otimes [[x, r_2], r'_1] \otimes r'_2 - r_1 \otimes [[x, r'_2], r_1] \otimes r'_2 \\
&+ r_1 \otimes r'_1 \otimes [[x, r_2], r'_2] - r_1 \otimes r'_1 \otimes [[x, r'_2], r_2] \\
&+ [r_1, r'_1] \otimes r_2 \otimes [x, r'_2] - [x, r_2] \otimes [r_1, r'_1] \otimes r'_2 - r_1 \otimes [x, r'_2] \otimes [r_2, r'_1] \\
&+ r_1 \otimes [r_2, r'_1] \otimes [x, r'_2] + [x, r_1] \otimes r'_1 \otimes [r_2, r'_2] + [r_1, r'_1] \otimes [x, r_2] \otimes r'_2.
\end{aligned}$$

These are exactly the twelve terms that we obtained on the left side times minus one. We only have to interchange the summation over the tensor components of r in some terms, i.e. we have to switch $r_1 \otimes r_2$ and $r'_1 \otimes r'_2$, which is allowed since these summations are all finite. Formula (3.3.3) is proved. Finally, the element δr is a coboundary, i.e. it fulfils the cocycle condition $\delta^2 r = 0$. Thus $(\mathfrak{g}, [\cdot, \cdot], \delta r)$ is a Lie bialgebra if and only if δr satisfies the coJacobi identity $\text{Alt}((\delta r \otimes 1)\delta r(x)) = 0$ for all $x \in \mathfrak{g}$. Now that Eq. (3.3.3) holds this is the case if and only if $\text{ad}_x^{(3)}(\text{CYB}(r)) = 0$ for all $x \in \mathfrak{g}$. This concludes the proof. \square

Definition 3.3.2 (Coboundary Lie Bialgebra) *The triple $(\mathfrak{g}, [\cdot, \cdot], r)$, where $r \in \mathfrak{g} \otimes \mathfrak{g}$, is said to be a coboundary Lie bialgebra if $(\mathfrak{g}, [\cdot, \cdot], \delta r)$ is a Lie bialgebra. In this case r is called a coboundary structure of $(\mathfrak{g}, [\cdot, \cdot], \delta r)$.*

Remark 3.3.3 By Theorem 3.3.1 $(\mathfrak{g}, [\cdot, \cdot], r)$ is a coboundary Lie bialgebra if and only if r is skew-symmetric and $\text{ad}_x^{(3)}(\text{CYB}(r)) = 0$. It is clear from the definition that a Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ can have more than one coboundary structure. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a coboundary structure of this coboundary Lie bialgebra and $\alpha \in \mathfrak{g} \wedge \mathfrak{g}$ is \mathfrak{g} -invariant, i.e. $\delta \alpha = 0$, also $r' = r + \alpha$ is a coboundary structure of $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ because $\delta r' = \delta r + 0 = \gamma$. If on the other hand r and r' are two coboundary structures of $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ there is $0 = \gamma - \gamma = \delta r - \delta r' = \delta(r - r')$, i.e. there is an \mathfrak{g} -invariant element $\alpha \in \mathfrak{g} \wedge \mathfrak{g}$ such that $r' = r + \alpha$. This gives a one-to-one correspondence between coboundary structures r' that are different from r and \mathfrak{g} -invariant elements $\alpha \in \mathfrak{g} \wedge \mathfrak{g}$ denoted by $(\Lambda^2 \mathfrak{g})^{\mathfrak{g}}$.

Following the spirit of this thesis we define the corresponding morphisms and substructures.

Definition 3.3.4 *Consider two coboundary Lie bialgebras $(\mathfrak{g}, [\cdot, \cdot], r)$ and $(\mathfrak{g}', [\cdot, \cdot]', r')$.*

*i.) A morphism of Lie bialgebras $\phi: (\mathfrak{g}, [\cdot, \cdot], r) \rightarrow (\mathfrak{g}', [\cdot, \cdot]', r')$ is said to be a **morphism of coboundary Lie bialgebras** if ϕ respects the coboundary structures, i.e.*

$$(\phi \otimes \phi)(r) = r'. \quad (3.3.5)$$

ii.) We call $(\mathfrak{h}, [\cdot, \cdot], r)$ a **coboundary Lie subbialgebra** of $(\mathfrak{g}, [\cdot, \cdot], r)$ if $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subbialgebra such that $r \in \bigwedge^2 \mathfrak{h}$.

What follows are several examples of types of coboundary structures of a Lie bialgebra. We start with the most general one.

Definition 3.3.5 (Quasitriangular Lie Bialgebra) A triple $(\mathfrak{g}, [\cdot, \cdot], r)$, where $r \in \mathfrak{g} \otimes \mathfrak{g}$, is called a **quasitriangular Lie bialgebra** if the following three conditions are satisfied,

- i.) $(\mathfrak{g}, [\cdot, \cdot], \delta r)$ is a Lie bialgebra,
- ii.) $\text{CYB}(r) = 0$,
- iii.) $r + \sigma(r)$ is \mathfrak{g} -invariant, i.e. $\text{ad}_x^{(2)}(r + \sigma(r)) = 0$ for all $x \in \mathfrak{g}$, where $\sigma: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the map that flips the tensor components.

We call $r \in \mathfrak{g} \otimes \mathfrak{g}$ a **quasitriangular structure** for a Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ if $\gamma = \delta r$ and conditions ii) and iii) are satisfied.

Quasitriangular Lie bialgebras are examples of coboundary Lie bialgebras. Indeed if $(\mathfrak{g}, [\cdot, \cdot], r)$ is a quasitriangular Lie bialgebra define $r' = \frac{1}{2}(r - \sigma(r))$ to be the skew-symmetric part of r . Because of condition iii.) in Definition 3.3.5 $\delta r = \delta r - \frac{1}{2}\delta(r + \sigma(r)) = \delta r'$. The skew-symmetric element r' satisfies

$$\text{ad}_x^{(3)}(\text{CYB}(r')) = 0 \quad (3.3.6)$$

and, for this reason it is a coboundary structure of $(\mathfrak{g}, [\cdot, \cdot], \delta r)$. Conversely, one can check that a coboundary Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], r')$ has a quasitriangular structure if and only if there is an element $T \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ such that

$$\text{CYB}(r') = \frac{1}{4}[T_{12}, T_{23}]. \quad (3.3.7)$$

In this case the quasitriangular structure can be chosen by $r = r' + \frac{1}{2}T$ (c.f. [39, Remark page 31]). A more specific type of coboundary Lie bialgebras is defined below. In the following chapters we are especially interested in this kind of Lie bialgebra.

Definition 3.3.6 (Triangular Lie Bialgebra) A coboundary Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], r)$ is said to be a **triangular Lie bialgebra** if $\text{CYB}(r) = 0$. Moreover an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is said to be a **triangular structure** on a Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ if the following three conditions are satisfied,

- i.) $\delta r = \gamma$,
- ii.) $\text{CYB}(r) = 0$,
- iii.) $r - \sigma(r) = 0$, i.e. r is skew-symmetric or $r \in \Lambda^2 \mathfrak{g}$.

In this case r is said to be a ***r*-matrix** of \mathfrak{g} . Condition ii.) is called the **classical Yang-Baxter equation**.

It follows from the definition above that triangular Lie bialgebras, and as a consequence also *r*-matrices, are in one-to-one correspondence to solutions of the classical Yang-Baxter equation in $\Lambda^2 \mathfrak{g}$. Furthermore, every triangular structure is quasitriangular and every triangular Lie bialgebra has also the structure of a quasitriangular Lie bialgebra. As a first example of triangular Lie bialgebras and *r*-matrices we go back to Example 3.2.5. There we endowed any two-dimensional noncommutative Lie algebra with a Lie bialgebra structure.

Example 3.3.7 Consider the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ over a field \mathbb{k} of characteristic zero with basis elements $X, Y \in \mathfrak{g}$ that satisfy $[X, Y] = X$. Then define

$$r = X \wedge Y = X \otimes Y - Y \otimes X. \quad (3.3.8)$$

We want to prove that this is an r -matrix on $(\mathfrak{g}, [\cdot, \cdot])$. The skew-symmetry is evident. Moreover r is a solution of the classical Yang-Baxter equation on \mathfrak{g} :

$$\begin{aligned} \text{CYB}(r) &= [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \\ &= [X, X] \otimes Y \otimes Y - [X, Y] \otimes Y \otimes X - [Y, X] \otimes X \otimes Y + [Y, Y] \otimes X \otimes X \\ &\quad + X \otimes [Y, X] \otimes Y - X \otimes [Y, Y] \otimes X - Y \otimes [X, X] \otimes Y + Y \otimes [X, Y] \otimes X \\ &\quad + X \otimes X \otimes [Y, Y] - X \otimes Y \otimes [Y, X] - Y \otimes X \otimes [X, Y] + Y \otimes Y \otimes [X, X] \\ &= -X \otimes Y \otimes X + X \otimes X \otimes Y - X \otimes X \otimes Y + Y \otimes X \otimes X \\ &\quad + X \otimes Y \otimes X - Y \otimes X \otimes X \\ &= 0. \end{aligned}$$

We recover the Lie bialgebra structure γ given in Example 3.2.5 because

$$\begin{aligned} \delta r(X) &= \text{ad}_X^{(2)}(r) = \text{ad}_X^{(2)}(X \otimes Y) - \text{ad}_X^{(2)}(Y \otimes X) \\ &= [X, X] \otimes Y + X \otimes [X, Y] - [X, Y] \otimes X - Y \otimes [X, X] \\ &= X \otimes X - X \otimes X \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \delta r(Y) &= \text{ad}_Y^{(2)}(r) = \text{ad}_Y^{(2)}(X \otimes Y) - \text{ad}_Y^{(2)}(Y \otimes X) \\ &= [Y, X] \otimes Y + X \otimes [Y, Y] - [Y, Y] \otimes X - Y \otimes [Y, X] \\ &= Y \otimes X - X \otimes Y \\ &= -X \wedge Y, \end{aligned}$$

i.e. $\gamma = \delta r$. Thus $(\mathfrak{g}, [\cdot, \cdot], r)$ is a triangular Lie bialgebra, r is a triangular structure on $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ and a r -matrix on \mathfrak{g} .

Remark 3.3.8 If $\phi: \mathfrak{g} \rightarrow \mathfrak{a}$ is a Lie algebra homomorphism to another Lie algebra \mathfrak{a} and r a triangular structures on \mathfrak{g} , then $(\phi \otimes \phi)r$ is a triangular structure on \mathfrak{a} . We prove this in Lemma 6.3.1. This is not true for an arbitrary coboundary structure. For example the image of a quasitriangular structure under a Lie algebra homomorphism does not need to be a quasitriangular structure (see e.g. [39, page 29]).

As promised before we pass from the infinitesimal world of Lie bialgebras to the global world of Poisson-Lie Groups, inspired by Theorem 3.2.8.

Proposition 3.3.9 Let $r \in \mathfrak{g} \wedge \mathfrak{g}$ be an r -matrix on \mathfrak{g} . The elements π_r^λ and π_r^ρ defined for $x \in \mathfrak{g}$ by

$$\pi_r^\lambda(x) = (T_e \lambda_x \otimes T_e \lambda_x) r \quad (3.3.9)$$

and

$$\pi_r^\rho(x) = (T_e \rho_x \otimes T_e \rho_x) r \quad (3.3.10)$$

are Poisson bivectors. Moreover,

$$\pi = \pi_r^\rho - \pi_r^\lambda \quad (3.3.11)$$

is a Poisson-Lie structure on G and it coincides with the one obtained in Theorem 3.2.8.

For a proof consider [39, Proposition 3.1] and [62, page 48]. We see that if $r \neq 0$ it follows that $\pi_r^\lambda(e) = \pi_r^\rho(e) = r \neq 0$. According to Eq. (3.2.30) π_r^λ and π_r^ρ are not Poisson-Lie structures while $\pi = \pi_r^\rho - \pi_r^\lambda$ is, according to the last theorem.

3.4 The Etingof-Schiffmann Subalgebra

In this section we discuss non-degeneracy of r -matrices. Remark that there is a one-to-one correspondence between non-degenerate r -matrices on \mathfrak{g} and non-degenerate 2-cocycles on \mathfrak{g} corresponding to the trivial representation of \mathfrak{g} on \mathbb{C} (c.f. [39, Proposition 3.3]). In other words, there is a symplectic structure on \mathfrak{g} if and only if there is a non-degenerate r -matrix on \mathfrak{g} . Thus the classification of non-degenerate r -matrices gets easier for small $H^2(\mathfrak{g}, \mathbb{C})$.

Proposition 3.4.1 *Let \mathfrak{g} be a complex semisimple Lie algebra. Then there are no non-degenerate triangular structures on \mathfrak{g} .*

PROOF: We follow [39, Proposition 5.2]. Assume there is a non-degenerate r -matrix $r \in \Lambda^2 \mathfrak{g}$ and consider the corresponding symplectic form $\omega \in \Lambda^2 \mathfrak{g}^*$. Now \mathfrak{g} is semisimple, then by Whitehead's Lemma (see Theorem 3.1.4) one has $H^2(\mathfrak{g}, \mathbb{C}) = \{0\}$, i.e. we find a 1-cocycle $f \in \mathfrak{g}^*$ such that

$$\omega(x, y) = \delta f(x, y) = f([x, y]) \quad (3.4.1)$$

for any $x, y \in \mathfrak{g}$. On the other hand, the Killing form $\kappa \in \Lambda^2 \mathfrak{g}^*$ is non-degenerate on any semisimple Lie algebra. One can define a flat map

$$\flat: \mathfrak{g} \ni x \mapsto \kappa(x, \cdot) \in \mathfrak{g}^* \quad (3.4.2)$$

with respect to κ with inverse map $\sharp: \mathfrak{g}^* \rightarrow \mathfrak{g}$. Let $x, y \in \mathfrak{g}$ be arbitrary and define $z = -f^\sharp$. Then

$$\omega(x, y) = f([x, y]) = \kappa(f^\sharp, [x, y]) = \kappa([x, y], z) \stackrel{(*)}{=} \kappa(x, [y, z]),$$

where we use the associativity of κ (see Proposition B.2.2 ii.) in $(*)$. If we set $y = z$ one has $\omega(x, z) = \kappa(x, [z, z]) = 0$ for all $x \in \mathfrak{g}$ and ω is degenerate, which is a contradiction. \square

However, one can always find a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that a r -matrix on \mathfrak{g} is non-degenerate viewed as an element of $\mathfrak{h} \wedge \mathfrak{h}$. For this we need the classical Yang-Baxter equation in coordinate form: let \mathfrak{g} be a finite-dimensional Lie algebra with basis $e_1, \dots, e_n \in \mathfrak{g}$. Since $[e_i, e_j] \in \mathfrak{g}$ there are unique numbers $C_{ij}^k \in \mathbb{k}$, called the *structure constants*, such that

$$[e_i, e_j] = \sum_{k=1}^n C_{ij}^k e_k, \quad (3.4.3)$$

for all $i, j \in \{1, \dots, n\}$. Now let $r = \frac{1}{2} \sum_{i,j=1}^n r^{ij} e_i \wedge e_j \in \mathfrak{g} \wedge \mathfrak{g}$ be a r -matrix.

Lemma 3.4.2 *The classical Yang-Baxter equation $\text{CYB}(r) = 0$ is equivalent to the equations*

$$\sum_{j,k=1}^n (r^{jm} r^{kl} C_{jk}^i + r^{ij} r^{mk} C_{jk}^l + r^{ij} r^{kl} C_{jk}^m) = 0, \quad (3.4.4)$$

where $i, m, l \in \{1, \dots, n\}$.

PROOF: Since $r^{ij} = -r^{ji} \in \mathbb{k}$ one has $r = \sum_{i,j=1}^n r^{ij} e_i \otimes e_j$. Then

$$\begin{aligned} [r_{12}, r_{23}] &= \left[\sum_{i,j=1}^n r^{ij} e_i \otimes e_j \otimes 1, \sum_{k,l=1}^n r^{kl} 1 \otimes e_k \otimes e_l \right] \\ &= \sum_{i,j,k,l=1}^n r^{ij} r^{kl} [e_i \otimes e_j \otimes 1, 1 \otimes e_k \otimes e_l] \\ &= \sum_{i,j,k,l=1}^n r^{ij} r^{kl} e_i \otimes [e_j, e_k] \otimes e_l \\ &= \sum_{i,j,k,l,m=1}^n r^{ij} r^{kl} C_{jk}^m e_i \otimes e_m \otimes e_l. \end{aligned}$$

Similarly one gets

$$\begin{aligned} [r_{13}, r_{23}] &= \sum_{i,j,k,l,m=1}^n r^{ij} r^{mk} C_{jk}^l e_i \otimes e_m \otimes e_l, \\ [r_{12}, r_{13}] &= \sum_{i,j,k,l,m=1}^n r^{jm} r^{kl} C_{jk}^i e_i \otimes e_m \otimes e_l. \end{aligned}$$

Thus

$$\begin{aligned} \text{CYB}(r) &= [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] \\ &= \sum_{i,j,k,l,m=1}^n (r^{jm} r^{kl} C_{jk}^i + r^{ij} r^{mk} C_{jk}^l + r^{ij} r^{kl} C_{jk}^m) e_i \otimes e_m \otimes e_l \end{aligned}$$

and the classical Yang-Baxter equation $\text{CYB}(r) = 0$ is equivalent to

$$\sum_{j,k=1}^n (r^{jm} r^{kl} C_{jk}^i + r^{ij} r^{mk} C_{jk}^l + r^{ij} r^{kl} C_{jk}^m) = 0 \text{ for all } i, m, l \in \{1, \dots, n\}. \quad \square$$

This result allows the following

Proposition 3.4.3 *Let $r \in \mathfrak{g} \wedge \mathfrak{g}$ be a r -matrix. Then the set*

$$\mathfrak{h}_r = \{(f \otimes 1)r \in \mathfrak{g} \mid f \in \mathfrak{g}^*\} \quad (3.4.5)$$

is a Lie subalgebra of \mathfrak{g} and $r \in \mathfrak{h}_r \wedge \mathfrak{h}_r \subseteq \mathfrak{g} \wedge \mathfrak{g}$ is non-degenerate in $\mathfrak{h}_r \wedge \mathfrak{h}_r$.

PROOF: We follow [39, Section 3.5]. The isomorphism $\mathbb{k} \otimes \mathfrak{g} \cong \mathfrak{g}$ implies $\mathfrak{h}_r \subseteq \mathfrak{g}$. Moreover, \mathfrak{h}_r is a Lie subalgebra of \mathfrak{g} since, if we take $f, g \in \mathfrak{g}^*$, then

$$\begin{aligned} \sum_{k=1}^n \text{ad}_{(f \otimes 1)r} e_k &= \sum_{k=1}^n [(f \otimes 1)r, e_k] = \sum_{k=1}^n \left[\sum_{i,j=1}^n r^{ij} f(e_i) e_j, e_k \right] \\ &= \sum_{i,j,k=1}^n r^{ij} f(e_i) [e_j, e_k] = \sum_{i,j,k,l=1}^n r^{ij} f(e_i) C_{jk}^l e_l. \end{aligned}$$

Similarly for g . Then we have

$$\begin{aligned} [(f \otimes 1)r, (g \otimes 1)r] &= \left[\sum_{i,j=1}^n r^{ij} f(e_i) e_j, \sum_{k,l=1}^n r^{kl} g(e_k) e_l \right] \\ &= \sum_{i,j,k,l=1}^n r^{ij} r^{kl} f(e_i) g(e_k) [e_j, e_l] \\ &= \sum_{i,j,k,l,m=1}^n r^{ij} r^{kl} C_{jl}^m f(e_i) g(e_k) e_m \\ &\stackrel{(*)}{=} \sum_{i,j,k,l,m=1}^n f(e_i) g(e_k) (-r^{jk} C_{jl}^i - r^{ij} C_{jl}^k) r^{lm} e_m \\ &= \sum_{i,j,k,l,m=1}^n (f(r^{kj} g(e_k) C_{jl}^i e_i) - g(r^{ij} f(e_i) C_{jl}^k e_k)) r^{lm} e_m \\ &= \sum_{l,m=1}^n (f(\text{ad}_{(g \otimes 1)r} e_l) - g(\text{ad}_{(f \otimes 1)r} e_l)) r^{lm} e_m \\ &= \sum_{l,m=1}^n ((f \circ \text{ad}_{(g \otimes 1)r}) \otimes 1 - (g \circ \text{ad}_{(f \otimes 1)r}) \otimes 1) r^{lm} e_l \otimes e_m \\ &= (h \otimes 1)r, \end{aligned}$$

where we used (3.4.4) (look at the second term and change $m \rightarrow k, k \rightarrow l, l \rightarrow m$) in $(*)$ and defined

$$h = (f \circ \text{ad}_{(g \otimes 1)r} - g \circ \text{ad}_{(f \otimes 1)r}) \in \mathfrak{g}^*.$$

The next step is to show that r is an element of $\mathfrak{h}_r \wedge \mathfrak{h}_r$. Since we already proved that $\mathfrak{h}_r \subseteq \mathfrak{g}$ is a Lie subalgebra we can choose a basis $e_1, \dots, e_k \in \mathfrak{g}$ of \mathfrak{h}_r , where $k \in \{1, \dots, n\}$, and complete this to a basis $e_1, \dots, e_n \in \mathfrak{g}$ of \mathfrak{g} . Thus $\mathfrak{h}_r = \text{span}_{\mathbb{k}}\{e_1, \dots, e_k\}$. For an arbitrary $f \in \mathfrak{g}^*$ this implies

$$(f \otimes 1)r = \sum_{i,j=1}^n r^{ij} f(e_i) e_j \in \text{span}_{\mathbb{k}}\{e_1, \dots, e_k\}. \quad (3.4.6)$$

Since $f \in \mathfrak{g}^*$ was arbitrary there must be $r^{ij} = 0$ for $j \notin \{1, \dots, k\}$. By the skew-symmetry of r^{ij} this implies $r^{ij} = 0$ for $i \notin \{1, \dots, k\}$ or $j \notin \{1, \dots, k\}$. Then indeed

$$r = \frac{1}{2} \sum_{i,j=1}^k r^{ij} e_i \wedge e_j \in \mathfrak{h}_r \wedge \mathfrak{h}_r. \quad (3.4.7)$$

We see that Eq. (3.4.6) actually reads $\sum_{i,j=1}^k r^{ij} f(e_i) e_j \in \text{span}_{\mathbb{k}}\{e_1, \dots, e_k\}$, and we have

$$\text{span}_{\mathbb{k}}\{e_1, \dots, e_k\} = \mathfrak{h}_r \subseteq \text{span}_{\mathbb{k}} \left\{ \sum_{j=1}^k r^{1j} e_j, \dots, \sum_{j=1}^k r^{kj} e_j \right\} \subseteq \text{span}_{\mathbb{k}}\{e_1, \dots, e_k\}. \quad (3.4.8)$$

In other words, the matrix $(r^{ij})_{ij} \in M_{k \times k}(\mathbb{k})$ has full rank. For this r is non-degenerate viewed as an element of $\mathfrak{h}_r \wedge \mathfrak{h}_r$. \square

Following a discussion [94] with J. Schnitzer, we do the following

Definition 3.4.4 (Etingof-Schiffmann Subalgebra) *The Lie subalgebra \mathfrak{h}_r defined in Eq. (3.4.5) is called Etingof-Schiffmann subalgebra for a r -matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$. The corresponding connected Lie group H_r is said to be the Etingof-Schiffmann subgroup for r .*

In Chapter 6 this is our tool to produce obstructions to twist star products on the sphere \mathbb{S}^2 .

Chapter 4

Twist Deformation

While the last chapter was dedicated to the understanding of Lie algebras with additional structure, also with some geometric aspects, we start by discussing pure algebraic features of general algebras here. The ideas and notions of Hopf algebras are developed in Appendix A and appear here the first time. A introduction to this topic is given in [70, Chapter 1]. Also consider [27], [72] and [100, Chapter 4]. In a way, Hopf algebras take over the role of the Lie group, like the algebra does for the manifold. By this we mean the following: in Chapter 2 we talked about Lie groups acting on manifolds and some of the involved properties. Now we define a process such that a Hopf algebra can act on an algebra. Of course there is no smooth structure. Instead, the algebraic relations of a Lie group action are taken as the new axioms. In this case the algebra is said to be a (left) Hopf algebra module. For our purpose this is still not enough, thus we force the module structure to be compatible with the algebra multiplication in addition. This leads to (left) Hopf algebra module algebras. But here comes the interesting question: what happens if we deform the Hopf algebra? Is this possible in a way such that the deformed Hopf algebra is still a Hopf algebra? And is there even a natural way to deform any (left) module algebra of this Hopf algebra such that the deformed algebra is a (left) module algebra for the deformed Hopf algebra? For a general deformation there is no reason for this to be the case, but for twist deformations there are positive answers to all of these questions and we give them in Section 4.1. We start by defining twists on a Hopf algebra and prove that they deform the Hopf algebra and any (left) module algebra of this Hopf algebra in the way we asked for. In a parallel move we evolve this theory for the inverse of a twist. This might sound exaggerated and quite trivial, but there are two reasons why we insist on this: first, the defining properties and deformations change slightly, thus, one has to be careful. And second, in literature both notions occur and it is reasonable to understand them both. As a last word to this topic we also prove that both definitions and the resulting deformations are indeed equivalent.

Since we are interested in star products we have to modify our definition of twist, i.e. we have to define twists on formal power series of universal enveloping algebras. Section 4.2 starts with a short repetition on universal enveloping algebras, their Hopf algebra structure and formal power series. We also discuss the problem that the resulting object is no Hopf algebra anymore in general. In this special situation we want to apply the results of Section 4.1. Instead of Hopf algebra twists we take those of the formal power series of universal enveloping algebras and instead of arbitrary algebras we take the algebra of smooth real-valued functions on a Poisson manifold with pointwise multiplication. If the deformed multiplication on the smooth functions is a star product on the manifold, this star product is said to be a twist star product. Moreover, one says that an arbitrary star product on a Poisson manifold can be induced by a twist if there is

a twist on the formal power series of any universal enveloping algebra such that the deformation of the pointwise multiplication via this twist gives the star product. Thus at the end of this chapter we arrive at our objects of interest: star products and in particular the question whether they can or can not be induced by a twist. As a last observation we prove that a twist on the formal power series of a universal enveloping algebra always induces a r -matrix on the underlying Lie algebra. We stress that this is no one-to-one correspondence. This is the most important connection to Chapter 3. In the next chapter we finally see the connection to homogeneous spaces.

4.1 Twists and left Hopf Algebra Module Algebras

We need the notion of Hopf algebras and refer to Appendix A for a conceptional motivation of the involved axioms (see Definition A.0.19). For instance, we use Sweedler's notation

$$\Delta(\xi) = \xi_{(1)} \otimes \xi_{(2)} \quad (4.1.1)$$

for the coproduct of a Hopf algebra element ξ , which is also discussed in detail in this appendix (consider Remark A.0.13). Do not mix this up with the short notation $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ of an arbitrary element \mathcal{F} of the tensor product of a Hopf algebra. Let $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ be a Hopf algebra and $\mathcal{F} \in H \otimes H$. We introduce some very useful abbreviations

$$\begin{aligned} \mathcal{F}_{12} &= \mathcal{F} \otimes 1_H, \\ \mathcal{F}_{23} &= 1_H \otimes \mathcal{F}, \\ \mathcal{F}_{13} &= \mathcal{F}_1 \otimes 1_H \otimes \mathcal{F}_2, \\ \mathcal{F}_{21} &= \mathcal{F}_2 \otimes \mathcal{F}_1 \otimes 1_H, \\ \mathcal{F}_{32} &= 1_H \otimes \mathcal{F}_2 \otimes \mathcal{F}_1, \\ \mathcal{F}_{31} &= \mathcal{F}_2 \otimes 1_H \otimes \mathcal{F}_1. \end{aligned}$$

All of them are elements in $H \otimes H \otimes H = H^{\otimes 3}$. Further, we denote by 1 the identity map $H \rightarrow H$ and by 1_H the unit of H . We define the central element in twist deformation, namely a twisting element, following [6, Section 3.1] and also adopt the notation that is used there. Similar references are [27, Section 7.8], [39, Section 9.5], and [44, Section 1].

Definition 4.1.1 (Drinfel'd Twist) *Let $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ be a Hopf algebra. An invertible element $\mathcal{F} \in H \otimes H$ is said to be a Drinfel'd twist or twist if the following two conditions are satisfied,*

$$i.) \mathcal{F}_{12}(\Delta \otimes 1)\mathcal{F} = \mathcal{F}_{23}(1 \otimes \Delta)\mathcal{F},$$

$$ii.) (\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1_H.$$

The first condition is said to be the 2-cocycle condition and the second one the normalization property.

Remark 4.1.2 There is also a kind of 2-cocycle condition

$$((\Delta \otimes 1)\mathcal{F}^{-1}) \cdot \mathcal{F}_{12}^{-1} = ((1 \otimes \Delta)\mathcal{F}^{-1}) \cdot \mathcal{F}_{23}^{-1} \quad (4.1.2)$$

for \mathcal{F}^{-1} , since \mathcal{F} is invertible. It is equivalent to Definition 4.1.1 i.), since we know that inverting reverses the order of elements and that the inverse element of $(\Delta \otimes 1)\mathcal{F}$ in $H^{\otimes 3}$ is $(\Delta \otimes 1)\mathcal{F}^{-1}$ and so on. Furthermore, \mathcal{F}^{-1} fulfils the normalization property

$$(\epsilon \otimes 1)\mathcal{F}^{-1} = (1 \otimes \epsilon)\mathcal{F}^{-1} = 1_H, \quad (4.1.3)$$

since $(\epsilon \otimes 1)\mathcal{F}^{-1} = (\epsilon \otimes 1)((\epsilon \otimes 1)\mathcal{F}\mathcal{F}^{-1}) = 1_H$ according to Definition 4.1.1 ii.). Later on we see that there is a Hopf algebra structure on H such that \mathcal{F}^{-1} is a twist in the sense of Definition 4.1.1.

The primary use of a twist \mathcal{F} is to deform the Hopf algebra structure. If we consider $\Delta^{\mathcal{F}}: H \rightarrow H \otimes H$ defined by

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F} \cdot \Delta(\xi) \cdot \mathcal{F}^{-1} \quad (4.1.4)$$

and $S^{\mathcal{F}}: H \rightarrow H$ defined by

$$S^{\mathcal{F}}(\xi) = U \cdot S(\xi) \cdot U^{-1}, \quad (4.1.5)$$

where $\xi \in H$, $m: H \otimes H \ni (\xi \otimes \zeta) \mapsto \xi \cdot \zeta \in H$ and $U = m((1 \otimes S)\mathcal{F})$. Now we can state the following

Theorem 4.1.3 *Let $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ be a Hopf algebra and \mathcal{F} a twist on H . Then one gets another Hopf algebra $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathbb{k})$, with the deformed coproduct and antipode defined in Eq. (4.1.4) and Eq. (4.1.5), respectively.*

PROOF: The proof we give is a mixture of the ones in [39, Proposition 9.6] and [70, Theorem 2.3.4]. We split the proof into three parts. First we prove that $(H, +, \Delta^{\mathcal{F}}, \epsilon, \mathbb{k})$ is a coalgebra (see Definition A.0.12). Since the algebra structure is not changed the second step is to check if the algebra and the coalgebra structure are compatible in the sense of a bialgebra (consider Definition A.0.17). Finally, $S^{\mathcal{F}}$ has to be an antipode to finish the proof. Thus the first question is whether the diagrams (A.0.29) commute. To prove that $\Delta^{\mathcal{F}}$ satisfies the coassociativity condition we need the coassociativity of Δ , the 2-cocycle condition of \mathcal{F} and the 2-cocycle condition of \mathcal{F}^{-1} . Furthermore, we use that Δ is an algebra map. Let $\xi \in H$. Then

$$\begin{aligned} (\Delta^{\mathcal{F}} \otimes 1)(\Delta^{\mathcal{F}}(\xi)) &= \mathcal{F}_{12} \cdot ((\Delta \otimes 1)(\mathcal{F} \cdot \Delta(\xi) \cdot \mathcal{F}^{-1})) \cdot \mathcal{F}_{12}^{-1} \\ &= \mathcal{F}_{12} \cdot ((\Delta \otimes 1)\mathcal{F}) \cdot ((\Delta \otimes 1)\Delta(\xi)) \cdot ((\Delta \otimes 1)\mathcal{F}^{-1}) \cdot \mathcal{F}_{12}^{-1} \\ &= \mathcal{F}_{23} \cdot ((1 \otimes \Delta)\mathcal{F}) \cdot ((1 \otimes \Delta)\Delta(\xi)) \cdot ((1 \otimes \Delta)\mathcal{F}^{-1}) \cdot \mathcal{F}_{23}^{-1} \\ &= \mathcal{F}_{23} \cdot ((1 \otimes \Delta)(\mathcal{F} \cdot \Delta(\xi) \cdot \mathcal{F}^{-1})) \cdot \mathcal{F}_{23}^{-1} \\ &= (1 \otimes \Delta^{\mathcal{F}})\Delta^{\mathcal{F}}(\xi). \end{aligned}$$

Then the normalization property of \mathcal{F} and \mathcal{F}^{-1} together with the fact that ϵ is an algebra map implies the second (and third) coalgebra axiom: for $\xi \in H$ we calculate

$$\begin{aligned} (\epsilon \otimes 1)(\Delta^{\mathcal{F}}(\xi)) &= ((\epsilon \otimes 1)(\mathcal{F})) \cdot ((\epsilon \otimes 1)(\Delta(\xi))) \cdot ((\epsilon \otimes 1)(\mathcal{F}^{-1})) \\ &= 1_H \\ &= (1 \otimes \epsilon)(\Delta^{\mathcal{F}}(\xi)). \end{aligned}$$

This means that $(H, +, \Delta^{\mathcal{F}}, \epsilon, \mathbb{k})$ is a coalgebra and the first step is done. According to Definition A.0.17 $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, \mathbb{k})$ is a bialgebra if $\Delta^{\mathcal{F}}$ and ϵ are algebra maps. Thus it remains to check this property for $\Delta^{\mathcal{F}}$. Let $\xi, \zeta \in H$. Then

$$\begin{aligned} \Delta^{\mathcal{F}}(\xi \cdot \zeta) &= \mathcal{F} \cdot \Delta(\xi \cdot \zeta) \cdot \mathcal{F}^{-1} \\ &= \mathcal{F} \cdot \Delta(\xi) \cdot \mathcal{F}^{-1} \cdot \mathcal{F} \cdot \Delta(\zeta) \cdot \mathcal{F}^{-1} \\ &= \Delta^{\mathcal{F}}(\xi) \cdot \Delta^{\mathcal{F}}(\zeta), \end{aligned}$$

since Δ is an algebra map. Also $\Delta^{\mathcal{F}}(1_H) = \mathcal{F} \cdot \Delta(1_H) \cdot \mathcal{F}^{-1} = 1_H \otimes 1_H$ holds. The third and last step is to check that $S^{\mathcal{F}}$ is an antipode of the bialgebra $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, \mathbb{k})$. First of all, remark that $U = m((1 \otimes S)\mathcal{F}) = \mathcal{F}_1 S(\mathcal{F}_2)$ is invertible since \mathcal{F} is invertible and $U^{-1} := m((S \otimes 1)\mathcal{F}^{-1}) = S(\mathcal{F}_1^{-1})\mathcal{F}_2^{-1}$ satisfies

$$\begin{aligned}
UU^{-1} &= \mathcal{F}_1 S(\mathcal{F}_2) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \\
&= \underbrace{\mathcal{F}_1^{-1} \epsilon(\mathcal{F}_2^{-1})}_{=1_H} \mathcal{F}_1 S(\mathcal{F}_2) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \\
&= \mathcal{F}_1^{-1} \mathcal{F}_1 S(\mathcal{F}_2) S(\mathcal{F}_1^{-1}) S((\mathcal{F}_2^{-1})_{(1)}) (\mathcal{F}_2^{-1})_{(2)} \mathcal{F}_2^{-1} \\
&= S((\mathcal{F}_2^{-1})_{(1)} \mathcal{F}_1^{-1} \mathcal{F}_2) (\mathcal{F}_2^{-1})_{(2)} \mathcal{F}_2^{-1} \\
&= m((m \otimes 1) \circ (1 \otimes S \otimes 1) (((1 \otimes \Delta)\mathcal{F}^{-1}) \cdot \mathcal{F}_{23}^{-1} \cdot \mathcal{F}_{12})) \\
&= m((m \otimes 1) \circ (1 \otimes S \otimes 1) ((\Delta \otimes 1)\mathcal{F}^{-1})) \\
&= (\mathcal{F}_1^{-1})_{(1)} S((\mathcal{F}_1^{-1})_{(2)}) \mathcal{F}_2^{-1} \\
&= \epsilon(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \\
&= 1_H,
\end{aligned}$$

where we used several times the normalization property of \mathcal{F}^{-1} and the antipode axiom for S and Δ as well as the 2-cocycle condition for \mathcal{F}^{-1} . Similarly

$$\begin{aligned}
U^{-1}U &= S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \mathcal{F}_1 S(\mathcal{F}_2) \\
&= S(\underbrace{\mathcal{F}_1 \epsilon(\mathcal{F}_2)}_{=1_H}) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \mathcal{F}_1 S(\mathcal{F}_2) \\
&= S(\mathcal{F}_1) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \mathcal{F}_1 \epsilon(\mathcal{F}_2) S(\mathcal{F}_2) \\
&= S(\mathcal{F}_1^{-1} \mathcal{F}_1) \mathcal{F}_2^{-1} \mathcal{F}_1 (\mathcal{F}_2)_{(1)} S((\mathcal{F}_2)_{(2)}) S(\mathcal{F}_2) \\
&= m((m \otimes 1) \circ (S \otimes 1 \otimes S) (1 \otimes \mathcal{F}_2^{-1} \mathcal{F}_1 (\mathcal{F}_2)_{(1)} \otimes \mathcal{F}_2 (\mathcal{F}_2)_{(2)})) \\
&= m((m \otimes 1) \circ (S \otimes 1 \otimes S) (\mathcal{F}_{12}^{-1} \mathcal{F}_{23} (1 \otimes \Delta)\mathcal{F})) \\
&= m((m \otimes 1) \circ (S \otimes 1 \otimes S) ((\Delta \otimes 1)\mathcal{F})) \\
&= S((\mathcal{F}_1)_{(1)}) (\mathcal{F}_1)_{(2)} S(\mathcal{F}_2) \\
&= \epsilon(\mathcal{F}_1) S(\mathcal{F}_2) \\
&= S(\epsilon(\mathcal{F}_1) \mathcal{F}_2) \\
&= S(1_H) \\
&= 1_H.
\end{aligned}$$

Now the antipode axiom for $S^{\mathcal{F}}$ and $\Delta^{\mathcal{F}}$ is not hard to verify. Let $\xi \in H$. Then

$$\begin{aligned}
m((S^{\mathcal{F}} \otimes 1)\Delta^{\mathcal{F}}(\xi)) &= m(US(\mathcal{F}_1 \xi_{(1)} \mathcal{F}_1^{-1})U^{-1} \otimes \mathcal{F}_2 \xi_{(2)} \mathcal{F}_2^{-1}) \\
&= US(\mathcal{F}_1^{-1}) S(\xi_{(1)}) S(\mathcal{F}_1) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \mathcal{F}_2 \xi_{(2)} \mathcal{F}_2^{-1} \\
&= US(\mathcal{F}_1^{-1}) S(\xi_{(1)}) \xi_{(2)} \mathcal{F}_2^{-1} \\
&= US(\mathcal{F}_1^{-1}) \epsilon(\xi) \mathcal{F}_2^{-1} \\
&= \epsilon(\xi) UU^{-1} \\
&= \eta(\epsilon(\xi)),
\end{aligned}$$

again by the antialgebra map property and antipode property of S and similarly

$$m((1 \otimes S^{\mathcal{F}})\Delta^{\mathcal{F}}(\xi)) = m(\mathcal{F}_1 \xi_{(1)} \mathcal{F}_1^{-1} \otimes US(\mathcal{F}_2 \xi_{(2)} \mathcal{F}_2^{-1})U^{-1})$$

$$\begin{aligned}
&= \mathcal{F}_1 \xi_{(1)} \mathcal{F}_1^{-1} \mathcal{F}_1 S(\mathcal{F}_2) S(\mathcal{F}_2^{-1}) S(\xi_{(2)}) S(\mathcal{F}_2) U^{-1} \\
&= \mathcal{F}_1 \xi_{(1)} \underbrace{S(\mathcal{F}_2^{-1} \mathcal{F}_2)}_{=1_H} S(\xi_{(2)}) S(\mathcal{F}_2) U^{-1} \\
&= \mathcal{F}_1 \xi_{(1)} S(\xi_{(2)}) S(\mathcal{F}_2) U^{-1} \\
&= \epsilon(\xi) \underbrace{\mathcal{F}_1 S(\mathcal{F}_2)}_{=U} U^{-1} \\
&= \eta(\epsilon(\xi)).
\end{aligned}$$

Then $S^{\mathcal{F}}$ is an antipode compatible to the bialgebra structure and $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathbb{k})$ is a Hopf algebra. \square

Assume that $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ is a cocommutative Hopf algebra, i.e. for $\xi \in H$ one has

$$\xi_{(2)} \otimes \xi_{(1)} = \xi_{(1)} \otimes \xi_{(2)} \quad (4.1.6)$$

or, equivalently, the commutativity of the following diagram

$$\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
& \searrow \Delta & \downarrow \sigma \\
& & H \otimes H,
\end{array} \quad (4.1.7)$$

where $\sigma: H \otimes H \rightarrow H \otimes H$ denotes the isomorphism flipping the tensor components called the *braiding isomorphism*. The deformed coproduct is not cocommutative in general, because for $\xi \in H$ one has

$$\sigma(\Delta^{\mathcal{F}}(\xi)) = \sigma(\mathcal{F} \cdot (\xi_{(1)} \otimes \xi_{(2)}) \cdot \mathcal{F}^{-1}) = \mathcal{F}_2 \xi_{(2)} \mathcal{F}_2^{-1} \otimes \mathcal{F}_1 \xi_{(1)} \mathcal{F}_1^{-1},$$

which in general does not coincide with

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F} \cdot \Delta(\xi) \cdot \mathcal{F}^{-1} = \mathcal{F}_1 \xi_{(1)} \mathcal{F}_1^{-1} \otimes \mathcal{F}_2 \xi_{(2)} \mathcal{F}_2^{-1}.$$

In this sense we can say that the twist \mathcal{F} sometimes also induces a quantization of the Hopf algebra.

Remark 4.1.4 We discovered in Remark 4.1.2 that conditions (4.1.2) and (4.1.3) for an invertible element $\mathcal{F}^{-1} \in H \otimes H$ are equivalent to the twist conditions for $(\mathcal{F}^{-1})^{-1} = \mathcal{F}$ stated in Definition 4.1.1. Let us look a bit closer at these conditions for \mathcal{F}^{-1} . Assume that \mathcal{F} is a twist. Then according to Theorem 4.1.3 there is the Hopf algebra structure $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathbb{k})$. In terms of this structure the twist conditions for \mathcal{F}^{-1} have the form

$$\begin{aligned}
\mathcal{F}_{12}^{-1} \cdot ((\Delta^{\mathcal{F}} \otimes 1) \mathcal{F}^{-1}) &= \mathcal{F}_{12}^{-1} \cdot (\mathcal{F} \Delta(\mathcal{F}_1^{-1}) \mathcal{F}^{-1} \otimes \mathcal{F}_2^{-1}) \\
&= \Delta(\mathcal{F}_1^{-1}) \mathcal{F}^{-1} \otimes \mathcal{F}_2^{-1} \\
&= ((\Delta \otimes 1) \mathcal{F}^{-1}) \cdot \mathcal{F}_{12}^{-1} \\
&= ((1 \otimes \Delta) \mathcal{F}^{-1}) \cdot \mathcal{F}_{23}^{-1} \\
&= \mathcal{F}_1^{-1} \otimes \Delta(\mathcal{F}_2^{-1}) \cdot \mathcal{F}^{-1} \\
&= \mathcal{F}_1^{-1} \otimes \mathcal{F}^{-1} \Delta^{\mathcal{F}}(\mathcal{F}_2^{-1})
\end{aligned}$$

$$= \mathcal{F}_{23}^{-1} \cdot ((1 \otimes \Delta^{\mathcal{F}}) \mathcal{F}^{-1})$$

and $(\epsilon \otimes 1) \mathcal{F}^{-1} = (1 \otimes \epsilon) \mathcal{F}^{-1} = 1_H$. Thus \mathcal{F}^{-1} is a twist on $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathbb{k})$ if \mathcal{F} is a twist on $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$. Arranging the above equations in another order we see that also the converse is true. Then \mathcal{F} is a twist on $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ if and only if \mathcal{F}^{-1} is a twist on $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathbb{k})$ and the convention chosen in Definition 4.1.1 is equivalent to the axiom system

$$((\Delta \otimes 1)J \cdot J_{12} = ((1 \otimes \Delta)J) \cdot J_{23}, \quad (4.1.8)$$

$$(\epsilon \otimes 1)J = (1 \otimes \epsilon)J = 1_H, \quad (4.1.9)$$

where we defined $J = \mathcal{F}^{-1}$. The procedure of twisting is even involutive, i.e. twisting the twisted structure gives back the original one. This can be seen quite easily, since for $\xi \in H$ one has

$$(\Delta^{\mathcal{F}})^{\mathcal{F}^{-1}}(\xi) = \mathcal{F}^{-1} \Delta^{\mathcal{F}}(\xi) (\mathcal{F}^{-1})^{-1} = \mathcal{F}^{-1} \mathcal{F} \Delta(\xi) \mathcal{F}^{-1} \mathcal{F} = \Delta(\xi) = (\Delta^{\mathcal{F}^{-1}})^{\mathcal{F}}(\xi)$$

and

$$\begin{aligned} (S^{\mathcal{F}})^{\mathcal{F}^{-1}}(\xi) &= \mathcal{F}_1^{-1} S(\mathcal{F}_2^{-1}) S^{\mathcal{F}}(\xi) S(\mathcal{F}_1) \mathcal{F}_2 \\ &= \mathcal{F}_1^{-1} S(\mathcal{F}_2^{-1}) \mathcal{F}_1 \underbrace{\mathcal{F}_1^{-1} \epsilon(\mathcal{F}_1^{-1})}_{=1_H} S(\mathcal{F}_2) S(\xi) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \underbrace{\mathcal{F}_2 \epsilon(\mathcal{F}_2)}_{=1_H} S(\mathcal{F}_1) \mathcal{F}_2 \\ &= \mathcal{F}_1^{-1} \epsilon(\mathcal{F}_1^{-1}) S(\mathcal{F}_2 \mathcal{F}_2^{-1}) S(\xi) S(\mathcal{F}_1 \mathcal{F}_1^{-1}) \mathcal{F}_2 \epsilon(\mathcal{F}_2) \\ &= S(\xi) \\ &= \mathcal{F}_1 S(\mathcal{F}_2) \mathcal{F}_1^{-1} \mathcal{F}_1 \epsilon(\mathcal{F}_1) S(\mathcal{F}_2^{-1}) S(\xi) S(\mathcal{F}_1) \mathcal{F}_2 \mathcal{F}_2^{-1} \epsilon(\mathcal{F}_2^{-1}) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \\ &= \mathcal{F}_1 S(\mathcal{F}_2) \mathcal{F}_1^{-1} S(\mathcal{F}_2^{-1}) S(\xi) S(\mathcal{F}_1) \mathcal{F}_2 S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \\ &= \mathcal{F}_1 S(\mathcal{F}_2) S^{\mathcal{F}^{-1}}(\xi) S(\mathcal{F}_1^{-1}) \mathcal{F}_2^{-1} \\ &= (S^{\mathcal{F}^{-1}})^{\mathcal{F}}(\xi). \end{aligned}$$

This is called *twisting back*. Thus it is not surprising that both conventions appear in the literature. We want to follow the \mathcal{F} -convention as it is done in [6], but remark that the J -convention is also very common. Consider for example the inspiring book [39] by P. Etingof and O. Schiffman.

In a next step we show that a twist \mathcal{F} on $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ also induces a deformation (quantization) on any left H -module algebra. Here and in the following we use the convention to denote $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ shortly by H and the deformed Hopf algebra $(H, +, \cdot, \eta, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}, \mathbb{k})$ by $H^{\mathcal{F}}$.

Definition 4.1.5 (Left Hopf Algebra Module Algebra) *Let $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ be a Hopf algebra and $(\mathcal{A}, \cdot, +, \mathbb{k})$ an algebra (see Definition A.0.10). Then $(\mathcal{A}, \cdot, +, \mathbb{k})$ is said to be a left H -module algebra if the following two conditions are satisfied.*

- i.) $(\mathcal{A}, \cdot, +, \mathbb{k})$ is a left H -module, i.e. there is a \mathbb{k} -linear map $\triangleright: H \otimes \mathcal{A} \rightarrow \mathcal{A}$ such that for all $\xi, \zeta \in H$ and $a \in \mathcal{A}$ one has

$$(\xi \zeta) \triangleright a = \xi \triangleright (\zeta \triangleright a) \quad (4.1.10)$$

and

$$1_H \triangleright a = a. \quad (4.1.11)$$

ii.) The H -module structure of $(\mathcal{A}, \cdot, +, \mathbb{k})$ respects the algebra multiplication, i.e. for all $\xi \in H$ and $a, b \in \mathcal{A}$ one has

$$\xi \triangleright (ab) = (\xi_{(1)} \triangleright a)(\xi_{(2)} \triangleright b) \quad (4.1.12)$$

and

$$\xi \triangleright 1_{\mathcal{A}} = \epsilon(\xi)1_{\mathcal{A}}. \quad (4.1.13)$$

If \mathcal{F} is a twist on H we get from Theorem 4.1.3 the deformed Hopf algebra $H^{\mathcal{F}}$. The question we are interested in is the following: is there a way to structure any left H -module algebra $(\mathcal{A}, \cdot, +, \mathbb{k})$ such that it becomes a left $H^{\mathcal{F}}$ -module? Is this possible in a way such that \mathcal{A} stays an algebra? And finally, is the new module structure compatible with the new multiplication? The next theorem gives a positive answer to all those questions.

Theorem 4.1.6 *Let $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ be a Hopf algebra and \mathcal{F} a twist on H . Moreover, consider a left H -module algebra $(\mathcal{A}, \cdot, +, \mathbb{k})$ and define a map $\star: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ for any $a, b \in \mathcal{A}$ by*

$$a \star b = m(\mathcal{F}^{-1} \triangleright (a \otimes b)) = (\mathcal{F}_1^{-1} \triangleright a)(\mathcal{F}_2^{-1} \triangleright b), \quad (4.1.14)$$

where $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the map $m(a \otimes b) = a \cdot b = ab$ and \triangleright the left H -module action of H on $(\mathcal{A}, \cdot, +, \mathbb{k})$. Remark that we extended the action to an action of $H \otimes H$ on $\mathcal{A} \otimes \mathcal{A}$ in the obvious way without changing the notation. Then $(\mathcal{A}, \star, +, \mathbb{k})$ is a left $H^{\mathcal{F}}$ -module algebra.

PROOF: We have to show that $(\mathcal{A}, \star, +, \mathbb{k})$ (which we will shortly denote by $\mathcal{A}^{\mathcal{F}}$ in the following) is an algebra. Moreover, we have to verify axioms i.) and ii.) of Definition 4.1.5, having in mind that \mathcal{A} is a left H -module algebra. First of all, $H^{\mathcal{F}}$ is a Hopf algebra according to Theorem 4.1.3. Next we prove that $\mathcal{A}^{\mathcal{F}}$ is an algebra. Let $a, b, c \in \mathcal{A}$ and calculate

$$\begin{aligned} (a \star b) \star c &= ((\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b)) \star c \\ &= \mathcal{F}_1^{-1} \triangleright ((\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b))(\mathcal{F}_2^{-1} \triangleright c) \\ &= ((\mathcal{F}_1^{-1})_{(1)} \triangleright (\mathcal{F}_1^{-1} \triangleright a)) \cdot ((\mathcal{F}_1^{-1})_{(2)} \triangleright (\mathcal{F}_2^{-1} \triangleright b)) \cdot (\mathcal{F}_2^{-1} \triangleright c) \\ &= (((\mathcal{F}_1^{-1})_{(1)} \mathcal{F}_1^{-1}) \triangleright a) \cdot (((\mathcal{F}_1^{-1})_{(2)} \mathcal{F}_2^{-1}) \triangleright b) \cdot (\mathcal{F}_2^{-1} \triangleright c) \\ &= m((m \otimes 1) \circ ((\mathcal{F}_1^{-1})_{(1)} \mathcal{F}_1^{-1} \otimes (\mathcal{F}_1^{-1})_{(2)} \mathcal{F}_2^{-1} \otimes \mathcal{F}_2^{-1}) \triangleright (a \otimes b \otimes c)) \\ &= m((m \otimes 1) \circ (((\Delta \otimes 1) \mathcal{F}^{-1}) \cdot \mathcal{F}_{12}^{-1}) \triangleright (a \otimes b \otimes c)) \\ &= m((m \otimes 1) \circ (((1 \otimes \Delta) \mathcal{F}^{-1}) \cdot \mathcal{F}_{23}^{-1}) \triangleright (a \otimes b \otimes c)) \\ &= m((m \otimes 1) \circ (\mathcal{F}_1^{-1} \otimes (\mathcal{F}_2^{-1})_{(1)} \mathcal{F}_1^{-1} \otimes (\mathcal{F}_2^{-1})_{(2)} \mathcal{F}_2^{-1}) \triangleright (a \otimes b \otimes c)) \\ &= (\mathcal{F}_1^{-1} \triangleright a) \cdot (((\mathcal{F}_2^{-1})_{(1)} \mathcal{F}_1^{-1}) \triangleright b) \cdot (((\mathcal{F}_2^{-1})_{(2)} \mathcal{F}_2^{-1}) \triangleright c) \\ &= (\mathcal{F}_1^{-1} \triangleright a) \cdot ((\mathcal{F}_2^{-1})_{(1)} \triangleright (\mathcal{F}_1^{-1} \triangleright b)) \cdot ((\mathcal{F}_2^{-1})_{(2)} \triangleright (\mathcal{F}_2^{-1} \triangleright c)) \\ &= (\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright ((\mathcal{F}_1^{-1} \triangleright b) \cdot (\mathcal{F}_2^{-1} \triangleright c))) \\ &= (\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright (b \star c)) \\ &= a \star (b \star c), \end{aligned}$$

where we used the 2-cocycle property (4.1.2) of \mathcal{F}^{-1} and the left H -module properties (4.1.10) and (4.1.12) of \mathcal{A} several times. This shows that the product \star is associative. There is even the same unit element $1_{\mathcal{A}}$ for \star . To see this, take $a \in \mathcal{A}$ and use properties (4.1.11), (4.1.13) and the normalization property (4.1.3) of \mathcal{F}^{-1} to get

$$a \star 1_{\mathcal{A}} = (\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright 1_{\mathcal{A}})$$

$$\begin{aligned}
&= (\mathcal{F}_1^{-1} \triangleright a) \cdot \epsilon(\mathcal{F}_2^{-1})1_{\mathcal{A}} \\
&= (\mathcal{F}_1^{-1} \epsilon(\mathcal{F}_2^{-1})) \triangleright a \\
&= 1_H \triangleright a \\
&= a \\
&= (\epsilon(\mathcal{F}_1^{-1})\mathcal{F}_2^{-1}) \triangleright a \\
&= (\epsilon(\mathcal{F}_1^{-1})1_{\mathcal{A}}) \cdot (\mathcal{F}_2^{-1} \triangleright a) \\
&= (\mathcal{F}_1^{-1} \triangleright 1_{\mathcal{A}}) \cdot (\mathcal{F}_2^{-1} \triangleright a) \\
&= 1_{\mathcal{A}} \star a.
\end{aligned}$$

Remark that also the \mathbb{k} -linearity of \triangleright was needed to prove the above formula. This is enough to show that $\mathcal{A}^{\mathcal{F}}$ is an algebra, since \triangleright is linear. The only problematic property of Definition 4.1.5 we have to check is (4.1.12). Let $a, b \in \mathcal{A}$ and $\xi \in H$. Then

$$\begin{aligned}
\xi \triangleright (a \star b) &= \xi \triangleright ((\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b)) \\
&= (\xi_{(1)} \triangleright (\mathcal{F}_1^{-1} \triangleright a)) \cdot (\xi_{(2)} \triangleright (\mathcal{F}_2^{-1} \triangleright b)) \\
&= m((\xi_{(1)}\mathcal{F}_1^{-1} \otimes \xi_{(2)}\mathcal{F}_2^{-1}) \triangleright (a \otimes b)) \\
&= m((\Delta(\xi) \otimes \mathcal{F}^{-1}) \triangleright (a \otimes b)) \\
&= m((\mathcal{F}^{-1} \cdot \Delta^{\mathcal{F}}(\xi)) \triangleright (a \otimes b)) \\
&= m(\mathcal{F}^{-1} \triangleright (\Delta^{\mathcal{F}}(\xi) \triangleright (a \otimes b))) \\
&= (\xi_{(1)}^{\mathcal{F}} \triangleright a) \star (\xi_{(2)}^{\mathcal{F}} \triangleright b),
\end{aligned}$$

where we use the short notation $\Delta^{\mathcal{F}}(\xi) = \xi_{(1)}^{\mathcal{F}} \otimes \xi_{(2)}^{\mathcal{F}}$. We applied Eq. (4.1.10) and Eq. (4.1.12) several times as well as the definition (4.1.4) of the deformed coproduct $\Delta^{\mathcal{F}}$. This proves the result. \square

This theorem tells us that we can deform the product of a left H -module algebra to obtain a left module algebra of the deformed Hopf algebra. Assume that $(\mathcal{A}, \cdot, +, \mathbb{k})$ is a commutative left H -module algebra, i.e. $ab = ba$ for all $a, b \in \mathcal{A}$. Then the deformed algebra $(\mathcal{A}, \star, +, \mathbb{k})$ (which we may shortly denote by $\mathcal{A}^{\mathcal{F}}$ as in the last proof) is not commutative again in general, since for $a, b \in \mathcal{A}$ the term

$$a \star b = (\mathcal{F}_1^{-1} \triangleright a)(\mathcal{F}_2^{-1} \triangleright b)$$

may not coincide with

$$b \star a = (\mathcal{F}_1^{-1} \triangleright b)(\mathcal{F}_2^{-1} \triangleright a).$$

Thus \mathcal{F} will not only deform but also quantize \mathcal{A} in a way which is compatible to the deformation (quantization) of H .

4.2 Twist Star Products

We are only interested in the case when H is a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a (finite-dimensional) Lie algebra \mathfrak{g} over \mathbb{k} . As a short repetition, the *universal enveloping algebra* of \mathfrak{g} is the tensor algebra of \mathfrak{g} modulo the relation

$$[\xi, \zeta] = \xi\zeta - \zeta\xi, \tag{4.2.1}$$

for all $\xi, \zeta \in \mathfrak{g}$. One structures $\mathcal{U}(\mathfrak{g})$ as a Hopf algebra by setting

$$\begin{aligned}\Delta_{\mathcal{U}(\mathfrak{g})}(\xi) &= \xi \otimes 1_{\mathbb{k}} + 1_{\mathbb{k}} \otimes \xi, \\ \epsilon_{\mathcal{U}(\mathfrak{g})}(\xi) &= 0, \\ S_{\mathcal{U}(\mathfrak{g})}(\xi) &= -\xi,\end{aligned}$$

where $\xi \in \mathfrak{g}$. This extends to algebra (anti) homomorphisms and yields a cocommutative Hopf algebra. In a general Hopf algebra H one calls an element $\xi \in H$ that satisfies $\Delta(\xi) = \xi \otimes 1_H + 1_H \otimes \xi$ *primitive*. We want to pass to formal power series thus we have to recap them first.

Let \mathbb{k} be a field of characteristic zero and define $K = \mathbb{k}[[\hbar]]$ to be the space of all sequences $a = (a_0, a_1, a_2, \dots)$ in \mathbb{k} that can be written as a formal power series

$$a = \sum_{r=0}^{\infty} \hbar^r a_r \quad (4.2.2)$$

with $a_r \in \mathbb{k}$ and *Planck's constant* \hbar . This space has the structure of an associative commutative unital ring if we declare a product on K by

$$ab = \left(\sum_{r=0}^{\infty} \hbar^r a_r \right) \left(\sum_{r=0}^{\infty} \hbar^r b_r \right) = \sum_{r=0}^{\infty} \hbar^r \left(\sum_{s=0}^r a_s b_{r-s} \right) \quad (4.2.3)$$

for all $a, b \in K$. If V is a \mathbb{k} -vector space, the formal power series $V[[\hbar]]$ form a K -module by defining for $a \in K$ and $v \in V[[\hbar]]$

$$av = \left(\sum_{r=0}^{\infty} \hbar^r a_r \right) \left(\sum_{r=0}^{\infty} \hbar^r v_r \right) = \sum_{r=0}^{\infty} \hbar^r \left(\sum_{s=0}^r a_s v_{r-s} \right) \quad (4.2.4)$$

with the same multiplication (4.2.3) for elements in $V[[\hbar]]$. The next step is to define a topology such that $V[[\hbar]]$ is complete. So choose a $C > 1$ and define a valuation on K by

$$\| a_n \hbar^n + a_{n+1} \hbar^{n+1} + \dots \| = C^{-n}, \quad (4.2.5)$$

where (a_1, a_2, \dots) is an arbitrary element of K such that a_n is the smallest coefficient that is not equal to zero. For the sequence $(0, 0, \dots)$ that is constant zero one defines the valuation to be 0. One can check that this is indeed a well-defined valuation on K called the *\hbar -adic valuation*. The topology induced by the \hbar -adic valuation is said to be the *\hbar -adic topology* and one can check that K is complete with respect to this topology. Moreover, if we extend the valuation to $V[[\hbar]]$, also $V[[\hbar]]$ is complete with respect to the \hbar -adic topology (c.f. [39, Section 1.1.1] and [105, Section 6.2.1]).

Coming back to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ we can now pass to $\mathcal{U}(\mathfrak{g})[[\hbar]]$. By extending $\Delta_{\mathcal{U}(\mathfrak{g})}$, $\epsilon_{\mathcal{U}(\mathfrak{g})}$ and $S_{\mathcal{U}(\mathfrak{g})}$ $\mathbb{k}[[\hbar]]$ -linearly one gets a Hopf algebra structure

$$(\mathcal{U}(\mathfrak{g})[[\hbar]], +, \cdot, \Delta, \epsilon, S, \mathbb{k})$$

on $\mathcal{U}(\mathfrak{g})[[\hbar]]$, too. Only the comultiplication $\Delta: \mathcal{U}(\mathfrak{g})[[\hbar]] \rightarrow \mathcal{U}(\mathfrak{g})[[\hbar]] \otimes \mathcal{U}(\mathfrak{g})[[\hbar]]$ has to be modified. To assure it is well-defined we have to take the completion $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ of $\mathcal{U}(\mathfrak{g})[[\hbar]] \otimes \mathcal{U}(\mathfrak{g})[[\hbar]]$, i.e. we have to assume

$$\Delta: \mathcal{U}(\mathfrak{g})[[\hbar]] \rightarrow (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]. \quad (4.2.6)$$

By the same argumentation twists on $\mathcal{U}(\mathfrak{g})[[\hbar]]$ are defined as elements of $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$. In complete analogy to the last section we define a twist for our special “Hopf algebra”.

Definition 4.2.1 Let \mathfrak{g} be a Lie algebra over \mathbb{k} . We call an element $\mathcal{F} \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ a *twist* on $\mathcal{U}(\mathfrak{g})[[\hbar]]$ if the following three conditions are satisfied,

- i.) $\mathcal{F}_{12}(\Delta \otimes 1)\mathcal{F} = \mathcal{F}_{23}(1 \otimes \Delta)\mathcal{F}$,
- ii.) $(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1_{\mathcal{U}(\mathfrak{g})[[\hbar]]}$,
- iii.) $\mathcal{F} = 1_{\mathcal{U}(\mathfrak{g})[[\hbar]]} \otimes 1_{\mathcal{U}(\mathfrak{g})[[\hbar]]} \pmod{\hbar}$.

Remark 4.2.2 One motivation for the additional axiom iii.) in Definition 4.2.1 is the following: if we consider a left $\mathcal{U}(\mathfrak{g})[[\hbar]]$ -module algebra \mathcal{A} , one gets a deformed product on \mathcal{A} via Eq. (4.1.14) by Theorem 4.1.6. Axiom iii.) assures that \star coincides with the original multiplication \cdot on \mathcal{A} in zero order of \hbar . If we think of quantization this is the classical limit that gives back the classical structure.

The main motivation to consider *twist products* \star , defined in Eq. (4.1.14), is that in many cases \star denotes a star product. We do not give a motivated introduction into the rich field of star products and formal deformation quantization, but refer to [30], [37, Chapter 3] and [105, Chapter 6]. For this thesis it suffices to see star products as products on formal power series of smooth functions of a manifold. We are mainly interested in their existence but not in their properties. Some nice examples of star products can be found in [85]. Remark that Kontsevich proved that there is a star product on any Poisson manifold (c.f. [61]).

Definition 4.2.3 (Star Product) A star product on a Poisson manifold (M, π) is a $\mathbb{R}[[\hbar]]$ -bilinear map

$$\star: \mathcal{C}^\infty(M)[[\hbar]] \times \mathcal{C}^\infty(M)[[\hbar]] \rightarrow \mathcal{C}^\infty(M)[[\hbar]] \quad (4.2.7)$$

of the form

$$f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g), \quad (4.2.8)$$

for $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$, where $C_r: \mathcal{C}^\infty(M)[[\hbar]] \times \mathcal{C}^\infty(M)[[\hbar]] \rightarrow \mathcal{C}^\infty(M)[[\hbar]]$ are $\mathbb{R}[[\hbar]]$ -bilinear bidifferential operators such that the following four conditions are satisfied,

- i.) \star is associative,
- ii.) it holds for $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$ that $C_0(f, g) = fg$,
- iii.) it holds for $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$ that $C_1(f, g) - C_1(g, f) = \{f, g\}$, where the Poisson bracket $\{\cdot, \cdot\}$ is defined in Eq. (3.2.21),
- iv.) it holds for $f \in \mathcal{C}^\infty(M)[[\hbar]]$ that $1 \star f = f = f \star 1$, where 1 denotes the function that is constant 1.

Condition ii.) of Definition 4.2.3 guarantees, that the classical limit $\hbar \rightarrow 0$ gives back the commutative pointwise multiplication of functions, while condition iii.) can be interpreted as follows: the star product \star *deforms* the Poisson bracket $\{\cdot, \cdot\}$ (or equivalently the Poisson bivector π corresponding to $\{\cdot, \cdot\}$). This is the correspondence principle we mentioned in the introduction. We only omitted a prefactor. If \star is a star product on a symplectic manifold (M, ω) one says that \star deforms the symplectic structure ω of M . This is nearby thinking of Eq. (3.2.23).

Inspired by Theorem 4.1.6 we have the following

Definition 4.2.4 (Twist Star Product) *Let (M, π) be a Poisson manifold and \star a star product on it. Then \star is said to be a star product induced by a twist or a twist star product if there is a Lie algebra \mathfrak{g} over \mathbb{R} together with a twist \mathcal{F} on $\mathcal{U}(\mathfrak{g})[[\hbar]]$ such that $\mathcal{C}^\infty(M)[[\hbar]]$ is a left $\mathcal{U}(\mathfrak{g})[[\hbar]]$ -module algebra via \triangleright and for all $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$ one has*

$$f \star g = m(\mathcal{F}^{-1} \triangleright (f \otimes g)) = (\mathcal{F}_1^{-1} \triangleright f)(\mathcal{F}_2^{-1} \triangleright g), \quad (4.2.9)$$

where m denotes the multiplication $\mathcal{C}^\infty(M)[[\hbar]] \otimes \mathcal{C}^\infty(M)[[\hbar]] \rightarrow \mathcal{C}^\infty(M)[[\hbar]]$.

There is a connection between twists on $\mathcal{U}(\mathfrak{g})[[\hbar]]$ and r -matrices on \mathfrak{g} . The correspondence principle states that the first order of the commutator of the star product gives the Poisson bracket of the manifold. In a very similar way a twist can be seen as a deformation of a r -matrix. Conversely, the skew-symmetrization of the first order of a twist gives a r -matrix. This is part of the following

Proposition 4.2.5 *Let \mathfrak{g} be a Lie algebra over \mathbb{R} and*

$$\mathcal{F} = \sum_{i=0}^{\infty} \hbar^i F_i \quad (4.2.10)$$

a twist on $\mathcal{U}(\mathfrak{g})[[\hbar]]$. Then

$$r := F_1^{-1} - \sigma(F_1^{-1}) \in \mathfrak{g} \wedge \mathfrak{g} \quad (4.2.11)$$

is a r -matrix on \mathfrak{g} , where σ denotes the braiding isomorphism.

PROOF: According to Definition 4.2.1 iii.) $F_0 = 1_{\mathcal{U}(\mathfrak{g})[[\hbar]]} \otimes 1_{\mathcal{U}(\mathfrak{g})[[\hbar]]} =: \mathbb{1}$ is an invertible element. It is known that in this case \mathcal{F} is invertible with inverse

$$\mathcal{F}^{-1} = \sum_{i=0}^{\infty} \hbar^i J_i, \quad (4.2.12)$$

where $J_0 = \mathbb{1}$ and $J_i = -\sum_{j=1}^i F_j J_{i-j}$ for $i > 0$. To choose J for the coefficients is natural having in mind Remark 4.1.4. Then r given in Eq. (4.2.11) is a well-defined object $r = J_1 - \sigma(J_1)$ in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. Let us write down the 2-cocycle condition (4.1.2) for \mathcal{F}^{-1} by using the multiplication rules of formal power series:

$$\sum_{i=0}^{\infty} \hbar^i \sum_{j=0}^i ((\Delta \otimes 1)J_j) \cdot (J_{i-j} \otimes \mathbb{1}) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=0}^i ((1 \otimes \Delta)J_j) \cdot (\mathbb{1} \otimes J_{i-j}). \quad (4.2.13)$$

The first order in \hbar reads

$$\begin{aligned} J_1 \otimes \mathbb{1} + (\Delta \otimes 1)J_1 &= ((\Delta \otimes 1)J_0) \cdot (J_1 \otimes \mathbb{1}) + ((\Delta \otimes 1)J_1) \cdot (J_0 \otimes \mathbb{1}) \\ &= ((1 \otimes \Delta)J_0) \cdot (\mathbb{1} \otimes J_1) + ((1 \otimes \Delta)J_1) \cdot (\mathbb{1} \otimes J_0) \\ &= \mathbb{1} \otimes J_1 + (1 \otimes \Delta)J_1. \end{aligned}$$

Since σ is an isomorphism also $\sigma(\mathcal{F}^{-1})$ fulfils Eq. (4.1.2) and the same calculation implies

$$\sigma(J_1) \otimes \mathbb{1} + (\Delta \otimes 1)\sigma(J_1) = \mathbb{1} \otimes \sigma(J_1) + (1 + \Delta)\sigma(J_1). \quad (4.2.14)$$

Subtracting Eq. (4.2.14) from $J_1 \otimes \mathbb{1} + (\Delta \otimes 1)J_1 = \mathbb{1} \otimes J_1 + (1 \otimes \Delta)J_1$ we get a condition on r , namely

$$r \otimes \mathbb{1} + (\Delta \otimes 1)r = \mathbb{1} \otimes r + (1 \otimes \Delta)r. \quad (4.2.15)$$

We first show that r is an element of $\mathfrak{g} \wedge \mathfrak{g}$. The skew-symmetry is clear, since $\sigma(r) = \sigma(J_1) - \sigma^2(J_1) = -r$. The question is whether r lies in the wedge product of \mathfrak{g} , i.e. if the first and second tensor component of r are primitive elements. With the help of Eq. (4.2.15) and the skew-symmetry of r we obtain

$$\begin{aligned} (\Delta \otimes 1)r - r_{23} - r_{13} &= r_{23} - r_{12} + (1 + \Delta)r - r_{23} - r_{13} \\ &= (1 + \Delta)r - r_{12} - r_{13} \\ &= -(1 + \Delta)\sigma(r) - r_{12} - r_{13} \\ &= -r_2 \otimes (r_1)_{(1)} \otimes (r_1)_{(2)} - r_{12} - r_{13}. \end{aligned}$$

Permuting the tensor factors as $(1, 2, 3) \rightarrow (2, 3, 1)$ gives no minus sign, since r is skew-symmetric. Thus

$$\begin{aligned} (\Delta \otimes 1)r - r_{23} - r_{13} &= -(r_1)_{(1)} \otimes (r_1)_{(2)} \otimes r_2 - r_{32} - r_{31} \\ &= -((\Delta + 1)r - r_{23} - r_{13}). \end{aligned}$$

But this implies $\Delta(r_1) \otimes r_2 = (\Delta + 1)r = r_{23} + r_{13} = \mathbb{1} \otimes r_1 \otimes r_2 + r_1 \otimes \mathbb{1} \otimes r_2$, i.e.

$$\Delta(r_1) = \mathbb{1} \otimes r_1 + r_1 \otimes \mathbb{1}. \quad (4.2.16)$$

Thus r_1 is a primitive element and for this $r_1 \in \mathfrak{g}$. Since r is skew-symmetric there must be $r_2 \in \mathfrak{g}$. This can also be seen by considering again Eq. (4.2.15) and using

$$(\Delta \otimes 1)r = r_{23} + r_{13}. \quad (4.2.17)$$

Then the condition for r_2 reads analogous to Eq. (4.2.17)

$$(1 \otimes \Delta)r = r_{13} + r_{12}. \quad (4.2.18)$$

Finally $r \in \mathfrak{g} \wedge \mathfrak{g}$. The only thing left to prove is that $\text{CYB}(r) = 0$. First remember that the Lie bracket on $\mathcal{U}(\mathfrak{g})$ is the commutator. Then we obtain with Eq. (4.2.17) and the skew-symmetry of r

$$\begin{aligned} \text{Alt}((r \otimes \mathbb{1})(\Delta \otimes 1)r) &= \text{Alt}(r_{12}(r_{23} + r_{13})) \\ &= \text{Alt}(r_1 \otimes r_2 r'_1 \otimes r'_2 + r_1 r'_1 \otimes r_2 \otimes r'_2) \\ &= r_1 \otimes r_2 r'_1 \otimes r'_2 + r_1 r'_1 \otimes r_2 \otimes r'_2 + r'_2 \otimes r_1 \otimes r_2 r'_1 + r'_2 \otimes r_1 r'_1 \otimes r_2 \\ &\quad + r_2 r'_1 \otimes r'_2 \otimes r_1 + r_2 \otimes r'_2 \otimes r_1 r'_1 \\ &= r_1 r'_1 \otimes r_2 \otimes r'_2 - r'_1 r_1 \otimes r_2 \otimes r'_2 + r_1 \otimes r_2 r'_1 \otimes r'_2 - r_1 \otimes r'_1 r_2 \otimes r'_2 \\ &\quad + r'_2 \otimes r_1 \otimes r_2 r'_1 - r'_2 \otimes r_1 \otimes r'_1 r_2 \\ &= [r_1, r'_1] \otimes r_2 \otimes r'_2 + r_1 \otimes [r_2, r'_1] \otimes r'_2 + r_1 \otimes r'_1 \otimes [r_2, r'_2] \\ &= [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \\ &= \text{CYB}(r), \end{aligned}$$

where we changed to summation several times. Now let us define the \mathbb{k} -linear maps

$$\text{Alt}_1: \mathcal{U}(\mathfrak{g})^{\otimes 3} \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes 3} \quad (4.2.19)$$

and

$$\text{Alt}_2: \mathcal{U}(\mathfrak{g})^{\otimes 3} \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes 3} \quad (4.2.20)$$

on factorizing tensors $a \otimes b \otimes c$ by $\text{Alt}_1(a \otimes b \otimes c) = c \otimes a \otimes b$ and $\text{Alt}_2 = \text{Alt}_1^2$. Clearly one has $\text{Alt}_1^3 = 1$. With this we can calculate

$$\begin{aligned} & ((\mathcal{F} - \sigma(\mathcal{F})) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)(\mathcal{F} - \sigma(\mathcal{F})) \\ &= (\mathcal{F} \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\mathcal{F} - (\sigma(\mathcal{F}) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\mathcal{F} \\ & - (\mathcal{F} \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\sigma(\mathcal{F}) + (\sigma(\mathcal{F}) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\sigma(\mathcal{F}) \\ &= (\mathcal{F} \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\mathcal{F} - \text{Alt}_1((\mathbb{1} \otimes \sigma(\mathcal{F})) \cdot (1 \otimes \Delta)\sigma(\mathcal{F})) \\ & - \text{Alt}_1((\mathbb{1} \otimes \mathcal{F}) \cdot (1 \otimes \Delta)\mathcal{F}) + (\sigma(\mathcal{F}) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\sigma(\mathcal{F}) \\ &= (\mathcal{F} \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\mathcal{F} - \text{Alt}_1((\sigma(\mathcal{F}) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\sigma(\mathcal{F})) \\ & - \text{Alt}_1((\mathcal{F} \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\mathcal{F}) + (\sigma(\mathcal{F}) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\sigma(\mathcal{F}) \\ &= \mathcal{A} - \text{Alt}_1(\mathcal{A}) + \mathcal{B} - \text{Alt}_1(\mathcal{B}), \end{aligned}$$

where we used the 2-cocycle condition for \mathcal{F} and $\sigma(\mathcal{F})$ and defined $\mathcal{A} = (\mathcal{F} \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\mathcal{F}$ and $\mathcal{B} = (\sigma(\mathcal{F}) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)\sigma(\mathcal{F})$. In this form the next step is quite easy:

$$\begin{aligned} & \text{Alt}(((\mathcal{F} - \sigma(\mathcal{F})) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)(\mathcal{F} - \sigma(\mathcal{F}))) \\ &= (1 + \text{Alt}_1 + \text{Alt}_2)((\mathcal{F} - \sigma(\mathcal{F})) \otimes \mathbb{1}) \cdot (\Delta \otimes 1)(\mathcal{F} - \sigma(\mathcal{F})) \\ &= (1 + \text{Alt}_1 + \text{Alt}_2)(\mathcal{A} - \text{Alt}_1(\mathcal{A}) + \mathcal{B} - \text{Alt}_1(\mathcal{B})) \\ &= \mathcal{A} - \text{Alt}_1(\mathcal{A}) + \mathcal{B} - \text{Alt}_1(\mathcal{B}) \\ & + \text{Alt}_1(\mathcal{A}) - \text{Alt}_2(\mathcal{A}) + \text{Alt}_1(\mathcal{B}) - \text{Alt}_2(\mathcal{B}) \\ & + \text{Alt}_2(\mathcal{A}) - \mathcal{A} + \text{Alt}_2(\mathcal{B}) - \mathcal{B} \\ &= 0. \end{aligned}$$

This equation has to be satisfied in every order of \hbar . In particular, for order \hbar^2 we get

$$0 = \text{Alt}((r \otimes \mathbb{1})(\Delta \otimes 1)r) = \text{CYB}(r) \quad (4.2.21)$$

by the above calculation. This concludes the proof. \square

Corollary 4.2.6 *Let \star be a twist star product on a Poisson manifold (M, π) . Then there is a r -matrix on a Lie algebra such that the formal power series of the corresponding universal enveloping algebra structure $\mathcal{C}^\infty(M)[[\hbar]]$ as a left module algebra.*

We connected the existence of a twist star product to solutions of the classical Yang-Baxter equation in the Lie algebra that bases the left module algebra structure of $\mathcal{C}^\infty(M)[[\hbar]]$.

Chapter 5

r -Matrices that lead to Homogeneous Structures

In this chapter we prove one of the main results of the thesis. Here we connect all previous chapters and justify their study. In a nutshell we show the following: if the Poisson bivector of a symplectic manifold is the image of a r -matrix under a Lie algebra action, the manifold has to be a homogeneous space. It is necessary to demand that the involved Lie algebra action integrates to a Lie group action. The manifolds we later consider are compact thus the integration condition is fulfilled naturally. Then an important step is to argue that this Lie group action is already locally transitive in this situation. Here the symplectic character of the manifold is essential. Remark that we develop parallel to the proceeding analogous results for the Etingof-Schiffmann subgroup of the r -matrix. It is surprising as well as of enormous use that one can restrict the Lie algebra to the Lie subalgebra in which the r -matrix is non-degenerate and still gets the desired result. Thinking of Section 2.3 and Section 2.4 there are many candidates for obstructions now: the higher pretzel surfaces are not homogeneous spaces and the 2-sphere can only be structured as a homogeneous space in very few ways. The question is whether all the requested assumptions are given for these examples. And moreover: why should it be desirable to find obstructions to Poisson bivectors that are images of r -matrices through Lie algebra homomorphisms? While Section 5.1 is fully dedicated to the proof of the essential theorem, Section 5.2 answers the latest question. The main motivation to consider such Poisson bivectors is the following: they occur naturally in the presence of twist star products. We have already shown in Proposition 4.2.5 that every twist on the formal power series of a universal enveloping algebra induces a r -matrix on the corresponding Lie algebra. If the twist star product deforms the symplectic structure of the manifold, the corresponding Poisson bivector is of the requested form. Thus we do not only give obstructions to special Poisson bivectors but to twist star products. Then this chapter ends with the statement that any connected compact symplectic manifold endowed with a twist star product can be structured as a homogeneous space. Following the way of the Etingof-Schiffmann subgroup we even know that there is a Lie group that acts transitively on the symplectic manifold such that the r -matrix is non-degenerate in the corresponding Lie algebra.

5.1 Symplectic Manifolds with a special Poisson Bivector

Suppose that (M, ω) is a symplectic manifold and assume that there is a r -matrix

$$r = \frac{1}{2} \sum_{i,j=1}^n r^{ij} e_i \wedge e_j \in \mathfrak{g} \wedge \mathfrak{g} \quad (5.1.1)$$

on a Lie algebra \mathfrak{g} with basis $\{e_1, \dots, e_n\}$ and a Lie algebra action

$$\phi: \mathfrak{g} \rightarrow \Gamma^\infty(TM) \quad (5.1.2)$$

of \mathfrak{g} on M such that for any $p \in M$ one has

$$\pi_p = \frac{1}{2} \sum_{i,j=1}^n r^{ij} \phi(e_i)_p \wedge \phi(e_j)_p. \quad (5.1.3)$$

If ϕ integrates to a Lie group action Φ , this group action is locally transitive. More precisely,

Lemma 5.1.1 *Let (M, ω) be a symplectic manifold with corresponding Poisson bivector π defined in Eq. (3.2.27). Assume there is a r -matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ and a Lie algebra action $\phi: \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ of \mathfrak{g} on M which integrates to a Lie group action $\Phi: G \times M \rightarrow M$ of G on M such that Eq. (5.1.3) holds. Then Φ is locally transitive. Furthermore, the restriction $\Phi|_{H_r}$ of Φ to $H_r \times M$, where H_r denotes the Etingof-Schiffmann subgroup corresponding to r , is locally transitive.*

PROOF: Fix $p \in M$ and choose an arbitrary $v_p \in T_p M$. Since $\tilde{\pi}: T_p^* M \rightarrow T_p M$ defined in Eq. (3.2.26) is surjective there is a $\alpha_p \in T_p^* M$ such that

$$v_p = -\pi_p(\cdot, \alpha_p) = \sum_{i,j=1}^n r^{ij} \alpha_p(\phi(e_i)_p) \phi(e_j)_p. \quad (5.1.4)$$

But this means that

$$v_p \in \text{span}_{\mathbb{K}} \left\{ \sum_{j=1}^n r^{1j} \phi(e_j)_p, \dots, \sum_{j=1}^n r^{nj} \phi(e_j)_p \right\} \subseteq \text{span}_{\mathbb{K}} \{ \phi(e_1)_p, \dots, \phi(e_n)_p \}. \quad (5.1.5)$$

Since $v_p \in T_p M$ was arbitrary the map

$$\phi|_p: \mathfrak{g} \ni \xi \mapsto \phi(\xi)_p \in T_p M$$

has to be surjective. For the last statement consider the Etingof-Schiffmann subalgebra \mathfrak{h}_r corresponding to r . Like in the proof of Proposition 3.4.3 we can choose a basis $\{e_1, \dots, e_k\}$ of $\mathfrak{h}_r \subseteq \mathfrak{g}$ and complete this to a basis $\{e_1, \dots, e_n\}$ of \mathfrak{g} , such that

$$r = \frac{1}{2} \sum_{i,j=1}^n r^{ij} e_i \wedge e_j = \frac{1}{2} \sum_{i,j=1}^k r^{ij} e_i \wedge e_j. \quad (5.1.6)$$

But this implies

$$\pi_p = \frac{1}{2} \sum_{i,j=1}^n r^{ij} \phi(e_i)_p \wedge \phi(e_j)_p = \frac{1}{2} \sum_{i,j=1}^k r^{ij} \phi(e_i)_p \wedge \phi(e_j)_p, \quad (5.1.7)$$

i.e. the restriction $\phi|_{\mathfrak{h}_r}|_p$ of $\phi|_p$ to \mathfrak{h}_r still satisfies the transitivity condition on $T_p M$ for every $p \in M$. Since ϕ was integrable this is also true for the restriction to a Lie subgroup. This gives $\Phi|_{H_r}: H_r \times M \rightarrow M$, the restriction of Φ to $H_r \times M$, which is locally transitive and we can restrict ourselves without loss of generality to the Etingof-Schiffmann subalgebra in which r is non-degenerate. \square

Our aim is to show that the setting of the last lemma already implies that M is a homogeneous space. For this we need another

Lemma 5.1.2 *Let $\Phi: G \times M \rightarrow M$ be a locally transitive Lie group action on a connected manifold M . Then there is only one orbit of Φ and it coincides with M .*

PROOF: Let ϕ be the Lie algebra action corresponding to the locally transitive Lie group action Φ . Fix $p \in M$ and denote by \mathfrak{g}_p the Lie algebra corresponding to G_p . Remark that this is possible since $G_p \subseteq G$ is a Lie group according to Theorem 2.1.1 ii.). The map $\beta_p: G/G_p \ni g \cdot G_p \mapsto \Phi(g, p) \in M$ is a diffeomorphism if Φ is transitive according to Theorem 2.2.6. Our aim is to show that β_p is also a diffeomorphism if Φ is just locally transitive. We first prove that β_p is a local diffeomorphism. We have already shown in Theorem 2.2.6 that β_p is well-defined and smooth. Thus we can differentiate β_p at $e \cdot G_p \in G/G_p$ and obtain

$$T_{e \cdot G_p} \beta_p: \mathfrak{g}/\mathfrak{g}_p \rightarrow T_{\beta_p(e \cdot G_p)} M = T_p M. \quad (5.1.8)$$

Then the map

$$\tilde{\phi}: \mathfrak{g}/\mathfrak{g}_p \ni [\xi] \mapsto \tilde{\phi}([\xi]) = T_{e \cdot G_p} \beta_p[\xi] = \phi(\xi)_p \in T_p M \quad (5.1.9)$$

is well-defined. Let $v_p \in T_p M$ be arbitrary. Since $\phi|_p: \mathfrak{g} \rightarrow T_p M$ is surjective there is a $\xi \in \mathfrak{g}$ such that $\phi(\xi)_p = v_p$. Then $\tilde{\phi}([\xi]) = \phi(\xi)_p = v_p$ and $\tilde{\phi}$ is surjective. To show that $\tilde{\phi}$ is also injective take $[\xi] \in \ker \tilde{\phi}$. For all $t \in \mathbb{R}$ one gets

$$\begin{aligned} \frac{d}{dt} \beta_p(\exp(t\xi) \cdot G_p) &= \frac{d}{dt} \Phi_{\exp(t\xi)}(p) \\ &= \left. \frac{d}{ds} \right|_{s=0} \Phi_{\exp(t\xi)}(\Phi_{\exp(s\xi)}(p)) \\ &= T_p \Phi_{\exp(t\xi)} \left. \frac{d}{ds} \right|_{s=0} \Phi_{\exp(s\xi)}(p) \\ &= T_p \Phi_{\exp(t\xi)} \left. \frac{d}{ds} \right|_{s=0} \beta_p(\exp(s\xi) \cdot G_p) \\ &= T_p \Phi_{\exp(t\xi)} \tilde{\phi}([\xi]) \\ &= 0. \end{aligned}$$

Since $\beta_p(\exp(0 \cdot \xi) \cdot G_p) = \Phi(e, p) = p$ the last calculation implies that $\beta_p(\exp(t\xi) \cdot G_p) = p$ for all $t \in \mathbb{R}$. This shows $\exp(t\xi) \in G_p$ for all $t \in \mathbb{R}$ and for this $\xi \in \mathfrak{g}_p$, thus $[\xi] = 0$ and $\tilde{\phi}$ is injective. Then for any $g \in G$

$$\beta_x(g \cdot G_p) = \Phi(g, p) = \Phi(g, \Phi(e, p)) = \Phi_g \circ \beta_p(e \cdot G_p) = \Phi_g \circ \beta_p \circ \ell_{g^{-1}}(g \cdot G_p) \quad (5.1.10)$$

implies

$$T_{g \cdot G_p} \beta_p = T_p \Phi_g \circ T_{e \cdot G_p} \beta_p \circ T_g \ell_{g^{-1}}, \quad (5.1.11)$$

where $\ell_{g^{-1}}: G/G_p \ni g' \cdot G_p \mapsto (g^{-1}g') \cdot G_p \in G/G_p$. Since Φ_g and ℓ_g^{-1} are diffeomorphisms also $T_{g \cdot G_p} \beta_p$ is bijective. Thus β_p is a local diffeomorphism for all $p \in M$, i.e. for any $g \cdot G_p \in G/G_p$ there is an open neighbourhood $U \subseteq G/G_p$ of $g \cdot G_p$ such that $\beta_p(U) \subseteq M$ is open. Choose any point $q \in G \cdot p$ of the orbit of p under Φ . Then for every $g \in \Phi_p^{-1}(\{q\}) \subseteq G$ there is an open neighbourhood $U \subseteq G/G_p$ such that $\beta_p(U) \subseteq G \cdot p \subseteq M$ is open. Since $q \in \beta_p(U)$ we have found an open neighbourhood of q that is contained in $G \cdot p$ and the orbit $G \cdot p$ is open. Let again $p \in M$ be arbitrary and assume that there is a point $q \in M \setminus \{G \cdot p\}$. Since M is always partitioned by the orbits of any group action one has $G \cdot p \cap G \cdot q = \emptyset$, since otherwise $G \cdot p = G \cdot q$ what is not possible because $q \notin G \cdot p$. By this we get a partition of M by open sets what gives a contradiction if M is connected. This concludes the proof. \square

Now we just collect our results and conclude one of the main theorems of this thesis.

Theorem 5.1.3 [15]. *Let (M, ω) be a connected symplectic manifold with corresponding Poisson bivector π , $r \in \mathfrak{g} \wedge \mathfrak{g}$ a r -matrix and $\phi: \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ a Lie algebra action of \mathfrak{g} on M that integrates to a Lie group action $\Phi: G \times M \rightarrow M$ of G on M such that for any $p \in M$ one has*

$$\pi_p = \frac{1}{2} \sum_{i,j=1}^n r^{ij} \phi(e_i)_p \wedge \phi(e_j)_p. \quad (5.1.12)$$

Then M can be structured as a homogeneous space via Φ . Moreover, the Etingof-Schiffmann subgroup H_r acts transitively on M .

PROOF: This is a direct consequence of Lemma 5.1.1 and Lemma 5.1.2. \square

For this, a geometric information provides topological impact on the manifold: if the Poisson bivector is the image of a r -matrix under an integrable Lie algebra action, the manifold has to be a homogeneous space. For instance, the Euler characteristic of such a compact space has to be non-negative according to Theorem 2.3.16. We make use of this in the next chapter to get obstructions on the higher pretzel surfaces. The great benefit of the second statement of Theorem 5.1.3 is that the r -matrix is non-degenerate in the Lie algebra of a Lie group that acts transitively on M . This means we have more information on an infinitesimal level. Later on this helps us to find obstructions in the case of $M = \mathbb{S}^2$.

It is clear that one can apply Theorem 5.1.3 to any leaf of a symplectic foliation of a Poisson manifold. Remember that for any Poisson manifold (M, π) the image of the sharp map

$$\sharp: T^*M \rightarrow TM \quad (5.1.13)$$

is a smooth distribution. By this we mean that $\text{im}(\sharp)$ is a set of linear subspaces $\text{im}(\sharp_x)$ of $T_x M$, where $x \in M$, such that each is spanned by a finite number of smooth vector fields $X_1, \dots, X_k \in \text{im}(\sharp)$, i.e. $\text{im}(\sharp_x) = \text{span}\{X_1(x), \dots, X_k(x)\}$. One can also imagine $\text{im}(\sharp_x)$ as the set

$$\{v \in T_x M \mid \exists f \in \mathcal{C}^\infty(M) \text{ such that } X_f(x) = v\}, \quad (5.1.14)$$

i.e. the set of tangent vectors which are images of Hamiltonian vector fields at x . Now $\text{im}(\sharp)$ integrates to a foliation of M , that is a cover of M by connected subsets which are said to be the leaves of this foliation. Moreover, the Poisson bivector π induces symplectic Poisson bivectors on these leaves and for this we obtain a foliation of M called a *symplectic foliation*. The leaves are maximal symplectic submanifolds of M which are said to be the *symplectic leaves* of M . All these facts can be found in [101, Chapter 2]. Thus we can modify our last result in the following way:

Corollary 5.1.4 *Let (M, π) be a Poisson manifold. Any symplectic leaf (S, π^S) of M whose Poisson bivector is the image of a r -matrix under an integrable Lie algebra action $\phi: \mathfrak{g} \rightarrow \Gamma^\infty(TS)$ is a homogeneous space. Moreover, the Etingof-Schiffmann subgroup of this r -matrix acts transitively on S .*

5.2 Structures induced by Twist Star Products

As an a posteriori motivation to consider Poisson bivectors of the form (5.1.3) we examine star products that are induced by a twist. In Corollary 4.2.6 we have seen the connection between twist star products and r -matrices. It is also interesting to consider the Poisson bracket and the Poisson bivector which are deformed by a twist star product.

Proposition 5.2.1 *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and \star a star product on M deforming the Poisson bracket $\{\cdot, \cdot\}$. Assume that \star is induced by a twist $\mathcal{F} = \sum_{i=0}^{\infty} \hbar^i F_i \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ and denote the left module action of $\mathcal{U}(\mathfrak{g})[[\hbar]]$ on $\mathcal{C}^\infty(M)[[\hbar]]$ by \triangleright . Then*

i.) *there is a r -matrix*

$$r = \sigma(F_1) - F_1 \quad (5.2.1)$$

on \mathfrak{g} ,

ii.) *the Poisson bracket reads*

$$\{f, g\} = m(r \triangleright (f \otimes g)) \quad (5.2.2)$$

for all $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$, where $m: \mathcal{C}^\infty(M)[[\hbar]] \otimes \mathcal{C}^\infty(M)[[\hbar]] \ni (f \otimes g) \mapsto fg \in \mathcal{C}^\infty(M)[[\hbar]]$,

iii.) *there is a Lie algebra action ϕ of \mathfrak{g} on M defined for any $\xi \in \mathfrak{g}$ by*

$$\mathcal{L}_{-\phi(\xi)} = \xi \triangleright: \mathcal{C}^\infty(M)[[\hbar]] \rightarrow \mathcal{C}^\infty(M)[[\hbar]], \quad (5.2.3)$$

where \mathcal{L} denotes the Lie derivative,

iv.) *the Poisson bivector π corresponding to $\{\cdot, \cdot\}$ reads*

$$\pi_p = \frac{1}{2} \sum_{i,j=1}^n r^{ij} \phi(e_i)_p \wedge \phi(e_j)_p \quad (5.2.4)$$

for any $p \in M$, where e_1, \dots, e_n is a basis of \mathfrak{g} and $r = \frac{1}{2} \sum_{i,j=1}^n r^{ij} e_i \wedge e_j$.

PROOF: For any $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$ one has

$$\begin{aligned} \sum_{i=0}^{\infty} \hbar^i C_i(f, g) &= f \star g \\ &= m(\mathcal{F}^{-1} \triangleright (f \otimes g)) \\ &= m\left(\sum_{i=0}^{\infty} \hbar^i F_i^{-1} \triangleright (f \otimes g)\right) \\ &= \sum_{i=0}^{\infty} \hbar^i m(((F_i^{-1})_1 \triangleright f) \otimes ((F_i^{-1})_2 \triangleright g)) \end{aligned}$$

$$= \sum_{i=0}^{\infty} \hbar^i ((F_i^{-1})_1 \triangleright f) \cdot ((F_i^{-1})_2 \triangleright g).$$

In particular, $C_1(f, g) = ((F_1^{-1})_1 \triangleright f) \cdot ((F_1^{-1})_2 \triangleright g) = m(F_1^{-1} \triangleright (f \otimes g))$. Then the corresponding Poisson bracket satisfies for all $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$

$$\begin{aligned} \{f, g\} &= C_1(f, g) - C_1(g, f) \\ &= m(F_1^{-1} \triangleright (f \otimes g)) - m(F_1^{-1} \triangleright (g \otimes f)) \\ &= m((F_1^{-1} - \sigma(F_1^{-1})) \triangleright (f \otimes g)) \\ &= m(r \triangleright (f \otimes g)), \end{aligned}$$

since pointwise multiplication is commutative, where r is the r -matrix on \mathfrak{g} defined in Eq. (4.2.11). Because $F_1^{-1} = -F_1$ we proved the first two statements. Now define a map

$$\phi: \mathfrak{g} \rightarrow \Gamma^\infty(TM) \quad (5.2.5)$$

via Eq. (5.2.3). Then ϕ is a Lie algebra action of \mathfrak{g} on M . To see this, choose $\xi, \zeta \in \mathfrak{g}$, $f \in \mathcal{C}^\infty(M)[[\hbar]]$ and calculate

$$\mathcal{L}_{[\phi(\xi), \phi(\zeta)]} f = [\mathcal{L}_{\phi(\xi)}, \mathcal{L}_{\phi(\zeta)}](f) = [-\xi \triangleright, -\zeta \triangleright](f) = [\xi, \zeta] \triangleright f = \mathcal{L}_{-\phi([\xi, \zeta])} f,$$

which implies $[\phi(\xi), \phi(\zeta)] = -\phi([\xi, \zeta])$, i.e. ϕ is a Lie algebra anti-homomorphism. This proves the third part. If we assume \mathfrak{g} to be finite-dimensional with basis $e_1, \dots, e_n \in \mathfrak{g}$ one has

$$r = \frac{1}{2} \sum_{i,j=1}^n r^{ij} e_i \wedge e_j = \sum_{i,j=1}^n r^{ij} e_i \otimes e_j, \quad (5.2.6)$$

and for this

$$\{f, g\} = \frac{1}{2} \sum_{i,j=1}^n r^{ij} (e_i \triangleright f)(e_j \triangleright g), \quad (5.2.7)$$

for all $f, g \in \mathcal{C}^\infty(M)[[\hbar]]$. As a consequence the Poisson bivector reads

$$\pi_p = \frac{1}{2} \sum_{i,j=1}^n r^{ij} \phi(e_i)_p \wedge \phi(e_j)_p, \quad (5.2.8)$$

for any $p \in M$. This concludes the proof. \square

We can use Theorem 5.1.3 to state a necessary condition under which a star product \star on a symplectic manifold (M, ω) can be induced by a twist: if \star deforms the Poisson bivector π corresponding to ω , it can only be induced by a twist \mathcal{F} if the diagram

$$\begin{array}{ccc} \mathcal{F}^{-1} & \xrightarrow{\triangleright} & \star \\ \text{1-st} \downarrow & & \downarrow \text{1-st} \\ r & \xrightarrow{\triangleright} & \pi \end{array}, \quad (5.2.9)$$

commutes. This results in the following

Theorem 5.2.2 *Let (M, ω) be a connected symplectic manifold endowed with a twist star product such that the Lie algebra action defined in Eq. (5.2.3) integrates to a Lie group action. Then M can be structured as a homogeneous space and the Etingof-Schiffmann subgroup corresponding to the r -matrix defined in Eq. (5.2.1) acts transitively on M .*

PROOF: This follows directly from Proposition 5.2.1 and Theorem 5.1.3. □

In particular, the requirements of Theorem 5.2.2 are fulfilled if the symplectic manifold is compact.

Corollary 5.2.3 *Let (M, ω) be a connected compact symplectic manifold endowed with a twist star product. Then M can be structured as a homogeneous space. Moreover, the Etingof-Schiffmann subgroup corresponding to the r -matrix defined in Eq. (5.2.1) acts transitively on M .*

PROOF: Since any flow of a vector field on a compact manifold is complete, a theorem of Palais (c.f. [71, Theorem 6.5]) implies that any Lie algebra action on a connected compact manifold integrates to a Lie group action. □

Chapter 6

Obstructions and Examples of Twist Star Products

In this chapter we gather all previous notions and results to prove or disprove the existence of twist star products deforming the symplectic structure on connected compact manifolds of dimension 2. One can generalize these results easily to arbitrary Poisson manifolds that inherit symplectic leaves of these types. It turns out that the examples that allow existence of twist star products deforming the symplectic structure are quite rare. In fact there is only one example of such a twist star product on the 2-torus and obstructions in all other cases.

Consider a symplectic manifold (M, ω) of dimension $\dim M = 2n \in \mathbb{N}$. Since $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$ is a non-degenerate 2-form on M the corresponding *Liouville form*

$$\Omega = \underbrace{\omega \wedge \dots \wedge \omega}_{n\text{-times}} \in \Gamma^\infty(\Lambda^{2n} T^*M) \quad (6.0.1)$$

is a non-degenerate $2n$ -form on M , i.e. M is orientable. For $n = 1$ the converse is also true: if M is a orientable 2-dimensional manifold, there is a non-degenerate 2-form $\Omega = \omega \in \Gamma^\infty(\Lambda^2 T^*M)$ on M . But since there are no 3-forms on a 2-dimensional manifold $d\omega = 0$, i.e. (M, ω) is a symplectic manifold. Thus a 2-dimensional manifold is symplectic if and only if it is orientable. There is a well-known classification of connected orientable compact 2-dimensional manifold: two such manifolds are diffeomorphic if and only if they have the same genus $g \in \mathbb{N}_0$ (c.f. [66, Section 12.1.5]). We denote by $T(g)$ an arbitrary representative of a connected orientable compact 2-dimensional manifold of genus g and call it the *pretzel surface* of genus g . The topological invariant g encodes the number of holes. Clearly $T(0) \cong \mathbb{S}^2$ is diffeomorphic to the 2-sphere and $T(1) \cong \mathbb{T}^2$ is diffeomorphic to the 2-torus. According to our discussion, the manifolds $T(g)$ are symplectic and on the other hand any 2-dimensional connected compact symplectic manifold is diffeomorphic to $T(g)$ for a unique $g \in \mathbb{N}_0$. We are also able to construct a generic symplectic form on $T(g)$. Since $T(g)$ is a manifold it is Hausdorff and second countable. Together with the property of being connected this implies the existence of a *Riemannian metric* $m \in \Gamma^\infty(S^2 T^*T(g))$ on $T(g)$, i.e. for any $p \in T(g)$

$$m_p: T_p T(g) \times T_p T(g) \rightarrow \mathbb{R} \quad (6.0.2)$$

is a positive definite symmetric bilinear form on $T_p T(g)$ (c.f. [2, Proposition 5.5.11]). In a positively oriented local chart $(U, x = (x_1, x_2))$ of $T(g)$ one has

$$m|_U = \sum_{i,j=1}^2 \frac{1}{2} m_{ij} dx^i \vee dx^j. \quad (6.0.3)$$

Then define

$$\omega|_U = \sum_{i,j=1}^2 \frac{1}{2} \sqrt{\det(m_{k\ell})_{k,\ell=1,2}} dx^i \wedge dx^j = \sqrt{m_{11}m_{22} - m_{12}m_{21}} dx^1 \wedge dx^2 \quad (6.0.4)$$

which gives a 2-form $\omega \in \Gamma^\infty(\Lambda^2 T^*T(g))$ on $T(g)$ by gluing together (6.0.4) on an atlas of $T(g)$.

6.1 The higher Pretzel Surfaces

According to [66, Section 12.1.4] the Euler characteristic of $T(g)$ is

$$\chi(T(g)) = 2 - 2g \quad (6.1.1)$$

for any $g \in \mathbb{N}_0$. In particular, one has $\chi(T(g)) < 0$ for $g > 1$. Then by Theorem 2.3.16 one can not structure $T(g)$ as a homogeneous space if $g > 1$. Corollary 5.2.3 immediately implies the following

Theorem 6.1.1 *Let $g > 1$. There is no twist star product on $T(g)$ deforming a symplectic structure of $T(g)$.*

On the other hand, there is a star product on any $T(g)$ obtained by the Fedosov construction since $T(g)$ is symplectic (c.f. [40]). Thus there are star products deforming the symplectic structure on $T(g)$ which can not be induced by a twist and we have infinitely many examples. This gives a partial proof of the main theorem stated in the introduction.

6.2 The 2-Torus

Surprisingly, there is a twist star product on the 2-torus \mathbb{T}^2 . Consider \mathbb{R}^2 with coordinates (x_1, x_2) . If we choose the zero Lie algebra \mathfrak{g} over \mathbb{R}^2 then $\mathcal{U}(\mathfrak{g}) = \mathbf{S}^\bullet \mathbb{R}^2$ can be endowed with a Hopf algebra structure $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for all $x \in \mathbb{R}^2$. As described in Section 4.2 this can be extended to a Hopf algebra $\mathbf{S}^\bullet \mathbb{R}^2[[\hbar]]$. Now define

$$\mathcal{F} = \exp(-i\hbar \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}) \in (\mathbf{S}^\bullet \mathbb{R}^2 \otimes \mathbf{S}^\bullet \mathbb{R}^2)[[\hbar]]. \quad (6.2.1)$$

This satisfies the twist conditions of Definition 4.2.1. Denote by \triangleright the standard action of \mathbb{R}^2 on \mathbb{T}^2 . Then

$$f \star g = m \circ \mathcal{F}^{-1} \triangleright (f \otimes g) = m(\exp(i\hbar \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}) \triangleright (f \otimes g)) \quad (6.2.2)$$

is the well-known Weyl-Moyal star product on \mathbb{T}^2 , where $f, g \in \mathcal{C}^\infty(\mathbb{T}^2)[[\hbar]]$. It deforms the symplectic structure on \mathbb{T}^2 . More information can be found in [75] and [77].

6.3 The 2-Sphere

Finally, we consider $T(0)$, i.e. the sphere \mathbb{S}^2 . It is also a connected compact symplectic manifold: take for example the symplectic form $\omega \in \Gamma^\infty(\Lambda^2 T^*\mathbb{S}^2)$ defined for any $x \in \mathbb{S}^2$ and $v, w \in T_x \mathbb{S}^2$ by

$$\omega_x(v, w) = x \cdot (v \times w), \quad (6.3.1)$$

where \cdot denotes the standard scalar product on \mathbb{R}^3 and \times the vector product. While the closedness of ω is trivially fulfilled, also the non-degeneracy can be concluded quite easily since $T_x\mathbb{S}^2$ is the plain of vectors that are perpendicular to x in \mathbb{R}^3 . Unfortunately, the trick used in Theorem 6.1.1 does not apply here since $\chi(\mathbb{S}^2) = 2 > 0$. But we have seen in Section 2.4 that there are only very few connected (non-equivalent) Lie groups that act transitively and effectively on \mathbb{S}^2 . The question is now the following: are the additional assumptions of being connected and effectiveness to specific to produce general obstructions? Anyway, the connected Lie groups that act transitively and effectively give only a few possibilities to structure \mathbb{S}^2 as a homogeneous space. But luckily for any transitive action on \mathbb{S}^2 one can construct a connected one that acts effective in addition, such that the r -matrix survives the transformation. As an immediate consequence of Corollary 5.2.3 we can assume that the Lie group that acts transitively is connected: if there is a twist star product on a compact symplectic manifold the Etingof-Schiffmann subgroup (a connected Lie group) corresponding to the r -matrix induced by the twist acts transitively on the manifold and the r -matrix is non-degenerate in the corresponding Lie algebra. We are interested in obstructions, thus this is exactly the implication we want since the existence of this connected Lie group acting transitively on \mathbb{S}^2 produces a contradiction in the end.

We now prove that r -matrices fit well in our categorial frame.

Lemma 6.3.1 *Let $\phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be a Lie algebra homomorphism. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a r -matrix then $\tilde{r} = (\phi \otimes \phi)r$ is a r -matrix on \mathfrak{h} .*

PROOF: This is a simple calculation:

$$\begin{aligned} \text{CYB}(\tilde{r}) &= [\phi(r_1), \phi(r'_1)]_{\mathfrak{h}} \otimes \phi(r_2) \otimes \phi(r'_2) + \phi(r_1) \otimes [\phi(r_2), \phi(r'_1)]_{\mathfrak{h}} \otimes \phi(r'_2) \\ &\quad + \phi(r_1) \otimes \phi(r'_1) \otimes [\phi(r_2), \phi(r'_2)]_{\mathfrak{h}} \\ &= \phi([r_1, r'_1]_{\mathfrak{g}}) \otimes \phi(r_2) \otimes \phi(r'_2) + \phi(r_1) \otimes \phi([r_2, r'_1]_{\mathfrak{g}}) \otimes \phi(r'_2) \\ &\quad + \phi(r_1) \otimes \phi(r'_1) \otimes \phi([r_2, r'_2]_{\mathfrak{g}}) \\ &= (\phi \otimes \phi \otimes \phi)\text{CYB}(r) \end{aligned}$$

by the Lie algebra homomorphism property of ϕ . Then, if r is a r -matrix one has $\text{CYB}(\tilde{r}) = (\phi \otimes \phi \otimes \phi)0 = 0$. Also the skew-symmetry of \tilde{r} is clear since ϕ is linear. \square

We pass from transitive to transitive and effective actions. With the help of the last lemma we do not loose the r -matrix.

Proposition 6.3.2 *Let G be a Lie group acting on a manifold M via Φ and define*

$$N = \{g \in G \mid \Phi_g = \text{id}_M\}. \quad (6.3.2)$$

If r is a r -matrix on the corresponding Lie algebra \mathfrak{g} then there is a r -matrix on the Lie algebra $\mathfrak{g}/\mathfrak{n}$ corresponding to G/N . If the action $\Phi: G \times M \rightarrow M$ is transitive, so is the effective action

$$\Psi: G/N \times M \ni (g \cdot N, x) \mapsto \Phi(g, x) \in M. \quad (6.3.3)$$

PROOF: We already know that N is a Lie subgroup of G . To see that it is a normal subgroup choose $g \in G$ and $n \in N$ and calculate

$$\Phi_{gng^{-1}} = \Phi_g \circ \Phi_n \circ \Phi_g^{-1} = \Phi_g \circ \Phi_g^{-1} = \text{id}_M,$$

which shows that $gng^{-1} \in N$. Thus $gNg^{-1} \subseteq N$ for all $g \in G$ and N is normal. It is well-known (c.f. [91, Proposition 5.7.8]) that the quotient G/H of a Lie group G and a normal subgroup $H \subseteq G$ is again a Lie group and there is a Lie group homomorphism $\pi: G \rightarrow G/H$. Consequently, we find a Lie group homomorphism $\pi: G \rightarrow G/N$. Then its tangent map $\phi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n}$ at $e \in G$ is a Lie algebra homomorphism. Thus the first claim of the proposition follows from Lemma 6.3.1. For the second one, we first check that Ψ defined in Eq. (6.3.3) is well-defined. Choose elements $g, g' \in G$ such that $g \cdot N = g' \cdot N$, i.e. there is a $h \in N$ such that $g' = gh$. Then

$$\Psi(g' \cdot N, x) = \Phi(g', x) = \Phi(g, \Phi(h, x)) = \Phi(g, x) = \Psi(g \cdot N, x)$$

for all $x \in M$. Also $\Psi(h \cdot N, x) = \Phi(h, x) = x$ for all $h \in N$. The smoothness as well as the action properties of Ψ are induced by Φ . It is effective by construction since the kernel N is eliminated. \square

Thus for our observations we can require without loss of generality that the Lie group acting transitively on the homogeneous space is connected and acts effectively in addition: if there is a r -matrix on \mathfrak{g} there also has to be one on $\mathfrak{g}/\mathfrak{n}$. For the sphere \mathbb{S}^2 we have classified all such actions in Section 2.4 up to the equivalence. Also this equivalence can be fixed.

Proposition 6.3.3 *Let G be a Lie group acting on M via Φ . If there is a r -matrix on \mathfrak{g} one can construct a r -matrix on the Lie algebra of any Lie group that acts in a to Φ equivalent way on M .*

PROOF: Let $\Phi: G \times M \rightarrow M$ and $\Phi': G' \times M \rightarrow M$ be two equivalent actions, i.e. there is a Lie group homomorphism $\phi: G \rightarrow G'$ such that $\Phi(g, x) = \Phi'(\phi(g), x)$ for all $g \in G, x \in M$. The Lie algebra action corresponding to ϕ maps a r -matrix on \mathfrak{g} to a r -matrix on \mathfrak{g}' according to Lemma 6.3.1. \square

Finally, we can prove the main theorem stated in the introduction.

Theorem 6.3.4 [15]. *There is no twist star product on \mathbb{S}^2 deforming any symplectic structure of \mathbb{S}^2 .*

PROOF: Assume there is a twist star product $\star = \sum_{i=0}^{\infty} \hbar^i C_i(\cdot, \cdot)$ on \mathbb{S}^2 such that

$$\{f, g\} = C_1(f, g) - C_1(g, f) \tag{6.3.4}$$

for all $f, g \in \mathcal{C}^\infty(\mathbb{S}^2)[[\hbar]]$. According to Proposition 5.2.1 there is a r -matrix $r' \in \mathfrak{g}' \wedge \mathfrak{g}'$ such that $\{f, g\} = m(r' \triangleright (f \otimes g))$ for all $f, g \in \mathcal{C}^\infty(\mathbb{S}^2)[[\hbar]]$ and the corresponding Lie group G' acts transitively on \mathbb{S}^2 . We proved in Proposition 6.3.2 that this implies the existence of a Lie group G which acts transitively and effectively on \mathbb{S}^2 and the existence of a r -matrix r on the corresponding Lie algebra \mathfrak{g} . Moreover, Corollary 5.2.3 states that the Etingof-Schiffmann subgroup H_r corresponding to r acts transitively on \mathbb{S}^2 . This action is the restriction of the action of G to $H_r \subseteq G$, which means that H_r acts effectively in addition. Since H_r is connected, it has to be a semisimple Lie group according to Lemma 2.4.5. Consequently, the Etingof-Schiffmann subalgebra \mathfrak{h}_r has to be a semisimple Lie algebra, in which $r \in \mathfrak{h}_r \wedge \mathfrak{h}_r$ is non-degenerate (consider Proposition 3.4.3). This is a contradiction to Proposition 3.4.1, since there we proved that there are no non-degenerate r -matrices on semisimple Lie algebras. \square

Remark that we did not use the concrete form of all non-equivalent connected Lie groups that act transitively and effectively on \mathbb{S}^2 (see Theorem 2.4.6) in Theorem 6.3.4. We also did not use Proposition 6.3.3 but only that the acting Lie groups are semisimple. We showed the existence of a connected Lie group, which is not semisimple, that acts transitively and effectively on \mathbb{S}^2 if the 2-sphere inherits a twist star product. Again, this result can be applied to any symplectic foliation of a Poisson manifold. In literature there are deformations of the q -deformed sphere (consider e.g. [25, 46, 65, 96]). These results are not contradictory to Theorem 6.3.4 since the corresponding Poisson structures are degenerate. But in this case the q -deformed sphere is not rotationally invariant, which is unfavorable if one wants to describe the symmetries of a physical system.

Appendix A

From Groups to Hopf Algebras

This appendix is meant to clarify notations, collect the most basic definitions of algebraic structures we use all the time, sometimes without referring to them, and prove some results that are of fundamental use. By decoding the corresponding axioms in commutative diagrams the process of dualizing objects is very nearby: it is immediately clear how the dual structures are defined since one just has to reverse arrows. By demanding compatibility to the dual notion we arrive at bialgebras. An interesting statement is that there are two equivalent definitions that are dual to each other. Then there is just one ingredient missing to define Hopf algebras, i.e. the antipode. It has some nice properties that we are going to prove. Of course we have a categorical approach in mind, thus we always define objects with their corresponding morphisms. We basically follow the first chapter of the book [70], but also refer to [88].

We start with arbitrary groups and how they can interact with other sets. A *group* is a non-empty set together with an associative multiplication and a unit element such that every element is invertible. The group is said to be *abelian* if the multiplication is commutative. A notion of groups that interact with other sets is

Definition A.0.5 (Group Action) *Let (G, \cdot) be a group and M an arbitrary set. We say that (G, \cdot) acts from the left on M if for every element $a \in G$ there is a map $M \rightarrow M$ denoted by $M \ni m \mapsto a \triangleright m \in M$ such that for all $a, b \in G$ and $m \in M$ one has*

$$(a \cdot b) \triangleright m = a \triangleright (b \triangleright m) \quad (\text{A.0.1})$$

and

$$e \triangleright m = m, \quad (\text{A.0.2})$$

where e denotes the unit of G . The axioms of a right action $\triangleleft: M \ni m \mapsto m \triangleleft a \in M$ read

$$m \triangleleft (a \cdot b) = (m \triangleleft a) \triangleleft b \text{ and } m \triangleleft e = m \quad (\text{A.0.3})$$

for all $a, b \in G$ and $m \in M$.

We often omit to write \triangleright or \triangleleft . As a first application we define vector spaces via group actions. This point of view gives much structural background to some axioms.

Definition A.0.6 (Vector Space) *Let $(\mathbb{k}, \cdot, +)$ be a field and $(V, +)$ an abelian group, where we denoted the addition on V also by $+$. The triple $(V, +, \mathbb{k})$ is said to be a vector space over the field \mathbb{k} if (\mathbb{k}, \cdot) acts on $(V, +)$ via \triangleright , i.e.*

$$(\lambda \cdot \mu) \triangleright v = \lambda \triangleright (\mu \triangleright v) \text{ and } 1 \triangleright v = v \quad (\text{A.0.4})$$

for all $\lambda, \mu \in \mathbb{k}$ and $v \in V$, where 1 denotes the unit of (\mathbb{k}, \cdot) and we set $0 \triangleright v = 0_V$ for all $v \in V$, where 0 denotes the unit of $(\mathbb{k}, +)$ and 0_V the unit of $(V, +)$. Moreover, these structures have to be compatible, i.e.

$$\lambda \triangleright (v + w) = \lambda \triangleright v + \lambda \triangleright w \text{ and } (\lambda + \mu) \triangleright v = \lambda \triangleright v + \mu \triangleright v \quad (\text{A.0.5})$$

for all $\lambda, \mu \in \mathbb{k}$ and $v, w \in V$. The map \triangleright , called scalar multiplication, is omitted in most cases.

We do not want to give an introduction to linear algebra, but recall the definitions of the tensor product and the dual space of a vector space since they are used in this thesis. In the following \mathbb{k} always denotes a field.

Example A.0.7 We start with the free abelian group on $V \times W$. This is the set of finite strings of tuples in $V \times W$ modulo the relation

$$(a, b) + (c, d) = (c, d) + (a, b), \quad (\text{A.0.6})$$

where $+$ denotes the concatenation of finite strings. We further quotient out for $\lambda \in \mathbb{k}$ and $(a, b), (c, d) \in V \times W$ all relations of the form

$$(\lambda a, b) = (a, \lambda b) \quad (\text{A.0.7})$$

$$(a + c, b) = (a, b) + (c, b) \quad (\text{A.0.8})$$

$$(a, b + d) = (a, b) + (a, d). \quad (\text{A.0.9})$$

The set we obtain is called the **tensor product** of V and W and is denoted by $V \otimes W$. We define a \mathbb{k} -action on the abelian group $(V \otimes W, +)$ for $\lambda \in \mathbb{k}$ and $(a, b) \in V \times W$ by

$$\lambda(a, b) = (\lambda a, b) \quad (\text{A.0.10})$$

which equals to $(a, \lambda b)$ in $V \otimes W$ according to Eq. (A.0.7) and extend this linearly to $V \otimes W$. Usually, one denotes tuples $(a, b) \in V \otimes W$ by $a \otimes b$. Indeed Eq. (A.0.10) defines a left action on $V \otimes W$, since for $\lambda, \mu \in \mathbb{k}$ and $a \otimes b \in V \otimes W$ one has

$$\begin{aligned} (\lambda\mu)(a \otimes b) &= (\lambda\mu a) \otimes b = \lambda(\mu a) \otimes b \\ &= \lambda(\mu(a \otimes b)) \end{aligned}$$

and $1(a \otimes b) = (1a) \otimes b = a \otimes b$. While for another $c \otimes d \in V \otimes W$ we have

$$\lambda(a \otimes b + c \otimes d) = \lambda(a \otimes b) + \lambda(c \otimes d)$$

by the linear extension and

$$\begin{aligned} (\lambda + \mu)(a \otimes b) &= ((\lambda + \mu)a) \otimes b = (\lambda a + \mu a) \otimes b \\ &= (\lambda a) \otimes b + (\mu a) \otimes b = \lambda(a \otimes b) + \mu(a \otimes b) \end{aligned}$$

and thus the triple $(V \otimes W, +, \mathbb{k})$ is a \mathbb{k} -vector space.

Remark that there is also a definition of tensor products via a universal property, consider e.g. [98]. As an example we want to discuss the \mathbb{k} -vector space $\mathbb{k} \otimes V$ for some \mathbb{k} -vector space V and realize that it is isomorphic to V itself. This identification is necessary for the definitions of algebras and coalgebras.

Lemma A.0.8 *Let V be a vector space over a field \mathbb{k} . Then $\mathbb{k} \otimes V \cong V \cong V \otimes \mathbb{k}$.*

PROOF: The isomorphisms we are searching for are

$$\mathbb{k} \otimes V \ni \lambda \otimes v \mapsto \lambda v \in V \quad (\text{A.0.11})$$

and

$$V \otimes \mathbb{k} \ni v \otimes \lambda \mapsto \lambda v \in V, \quad (\text{A.0.12})$$

where the right hand side denotes the \mathbb{k} -action on the \mathbb{k} -vector space V . We prove that the map defined in Eq. (A.0.11) is an isomorphism. The proof for the second isomorphism works exactly the same. The map defined in Eq. (A.0.11) is surjective, since any $v \in V$ can be produced via $1 \otimes v \mapsto 1v = v$. For injectivity assume there are $\lambda \otimes v, \mu \otimes w \in \mathbb{k} \otimes V$ such that $\lambda v = \mu w$. If $\lambda = 0$ then $\mu w = 0$, i.e. $\mu = 0$ or $w = 0$ and for this $\mu \otimes w = 0 = \lambda \otimes v$ because

$$0 \otimes w = (0 \cdot 0) \otimes w = 0 \otimes (0 \cdot w) = 0 \otimes 0 = 0.$$

Thus assume that $\lambda \neq 0$. Then $v = \lambda^{-1}\mu w$ and

$$\begin{aligned} \lambda \otimes v &= \lambda \otimes (\lambda^{-1}\mu w) = (\lambda\lambda^{-1}\mu) \otimes w \\ &= \mu \otimes w. \end{aligned}$$

This completes the proof. □

Now we discuss the last vector space example.

Example A.0.9 The set V^* of all \mathbb{k} -linear maps $V \rightarrow \mathbb{k}$ is called the **dual vector space** of the \mathbb{k} -vector space $(V, +, \mathbb{k})$. Thus an element $\phi \in V^*$ is a map $\phi: V \rightarrow \mathbb{k}$ such that for all $\lambda, \mu \in \mathbb{k}$ and $v, w \in V$

$$\phi(\lambda v + \mu w) = \lambda\phi(v) + \mu\phi(w). \quad (\text{A.0.13})$$

Then V^* becomes a \mathbb{k} -vector space structure if we define an addition on V^* pointwise for two elements $\phi, \psi \in V^*$ by

$$(\phi + \psi)(v) = \phi(v) + \psi(v), \quad (\text{A.0.14})$$

for all $v \in V$, where the right hand side denotes the addition in \mathbb{k} and if we define an action of \mathbb{k} on V^* pointwise for $\lambda \in \mathbb{k}$ and $\phi \in V^*$ by

$$(\lambda\phi)(v) = \phi(\lambda v), \quad (\text{A.0.15})$$

where the argument of ϕ in the right hand side is the \mathbb{k} -action of λ on v . The vector space axioms are easy to check. Note that there is a way to identify an element $v \in V$ with an element in the dual space V^{**} of V^* by defining $v(\phi) := \phi(v) \in \mathbb{k}$ for any $\phi \in V^*$. We proved

$$V \subseteq V^{**}. \quad (\text{A.0.16})$$

Moreover, if $\phi \in V^*$ and $\psi \in W^*$ we define a map $\phi \otimes \psi \in V^* \otimes W^*$ by $(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w)$ for $v \in V, w \in W$. Remark that by Lemma A.0.8 one has $\phi(v) \otimes \psi(w) \in \mathbb{k} \otimes \mathbb{k} \cong \mathbb{k}$ thus $\phi \otimes \psi: V \otimes W \rightarrow \mathbb{k}$. We calculate for $\lambda, \mu \in \mathbb{k}$ and $v \otimes w, x \otimes y \in V \otimes W$

$$(\phi \otimes \psi)(\lambda(v \otimes w) + \mu(x \otimes y)) = (\phi \otimes \psi)((\lambda v) \otimes w) + (\phi \otimes \psi)((\mu x) \otimes y)$$

$$\begin{aligned}
&= \phi(\lambda v) \otimes \psi(w) + \phi(\mu x) \otimes \psi(y) \\
&= (\lambda \phi(v)) \otimes \psi(w) + (\mu \phi(x)) \otimes \psi(y) \\
&= \lambda(\phi(v) \otimes \psi(w)) + \mu(\phi(x) \otimes \psi(y)) \\
&= \lambda((\phi \otimes \psi)(v \otimes w)) + \mu((\phi \otimes \psi)(x \otimes y))
\end{aligned}$$

what shows $\phi \otimes \psi \in (V \otimes W)^*$. Thus we proved

$$V^* \otimes W^* \subseteq (V \otimes W)^*. \quad (\text{A.0.17})$$

In finite dimensions one can check that Eq. (A.0.16) and Eq. (A.0.17) are equalities.

We are now able to define a very central object in Hopf algebra theory, first in terms of rings, fields and actions: an algebra.

Definition A.0.10 (Algebra) *We call $(\mathcal{A}, \cdot, +, \mathbb{k})$ an algebra over a field \mathbb{k} if the following three conditions are satisfied,*

- i.) the triple $(\mathcal{A}, \cdot, \mathbb{k})$ is a unital ring,*
- ii.) the triple $(\mathcal{A}, +, \mathbb{k})$ is a vector space,*
- iii.) the scalar multiplication (that exists according to ii.) is compatible with both, the multiplication and the addition, i.e. for all $\lambda \in \mathbb{k}$ and $a, b \in \mathcal{A}$ one has*

$$\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b) \quad (\text{A.0.18})$$

and

$$\lambda(a + b) = \lambda a + \lambda b \quad (\text{A.0.19})$$

while the second condition always holds since $(\mathcal{A}, +, \mathbb{k})$ is a vector space according to ii.).

We also want to give an alternative definition of an algebra. Remark that conditions (A.0.18) and (A.0.19) can be formulated as follows: there is a \mathbb{k} -linear map

$$\cdot : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \quad (\text{A.0.20})$$

replacing the multiplication on \mathcal{A} . If we assume in addition that $(\mathcal{A}, +, \mathbb{k})$ is a vector space the only conditions for $(\mathcal{A}, \cdot, +, \mathbb{k})$ to be an algebra are conditions on the new multiplication defined in Eq. (A.0.20). More precisely, the map defined in Eq. (A.0.20) has to obey for all $a, b, c \in \mathcal{A}$

$$(\cdot \circ (\cdot \otimes 1))(a \otimes b \otimes c) = (\cdot \circ (1 \otimes \cdot))((a \otimes b \otimes c)), \quad (\text{A.0.21})$$

where $1: \mathcal{A} \rightarrow \mathcal{A}$ denotes the identity map on \mathcal{A} , i.e. \cdot has to be associative and there has to be an element $1_{\mathcal{A}} \in \mathcal{A}$ such that

$$\cdot(a \otimes 1_{\mathcal{A}}) = a = \cdot(1_{\mathcal{A}} \otimes a), \quad (\text{A.0.22})$$

i.e. there has to be a unit $1_{\mathcal{A}}$ on \mathcal{A} . Another way to write condition (A.0.22) is obtained in the following way. Define for any $a \in \mathcal{A}$ a map

$$\eta_a: \mathbb{k} \rightarrow \mathcal{A} \quad (\text{A.0.23})$$

by $\eta_a(\lambda) = \lambda a$, where the right hand side denotes the scalar multiplication. Finally, set $\eta = \eta_{1_{\mathcal{A}}}$ what means

$$\eta(\lambda) = \lambda 1_{\mathcal{A}}. \quad (\text{A.0.24})$$

Thus a condition recovering (A.0.22) is

$$(\cdot \circ (\eta \otimes 1))(\lambda \otimes a) = \lambda a = (\cdot \circ (1 \otimes \eta))(a \otimes \lambda), \quad (\text{A.0.25})$$

for $\lambda \in \mathbb{k}$ and $a \in \mathcal{A}$. Conditions (A.0.21) and (A.0.25) are equivalent to the commutativity of the following diagrams, respectively.

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{1 \otimes \cdot} & \mathcal{A} \otimes \mathcal{A} \\ \cdot \otimes 1 \downarrow & & \downarrow \cdot \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\cdot} & \mathcal{A} \end{array} \quad \begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & & \\ \eta \otimes 1 \uparrow & \searrow \cdot & \\ \mathbb{k} \otimes \mathcal{A} = \mathcal{A} & & \end{array} \quad \begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & & \\ 1 \otimes \eta \uparrow & \searrow \cdot & \\ \mathcal{A} \otimes \mathbb{k} = \mathcal{A} & & \end{array} \quad (\text{A.0.26})$$

Remark that in the second and third diagram of (A.0.26) the equality sign denotes the isomorphisms defined in Eq. (A.0.11) and Eq. (A.0.12) of Lemma A.0.8, respectively. In the above lines we proved the following

Proposition A.0.11 *$(\mathcal{A}, \cdot, +, \mathbb{k})$ is an algebra over a field \mathbb{k} if and only if $(\mathcal{A}, +, \mathbb{k})$ is a \mathbb{k} -vector space and the following two conditions are satisfied.*

- i.) *There is a \mathbb{k} -linear map $\cdot: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, called **multiplication** on \mathcal{A} that is **associative**, i.e. the first diagram of (A.0.26) commutes.*
- ii.) *There is the \mathbb{k} -linear map η defined in Eq. (A.0.24) such that the second and third diagram of (A.0.26) commute.*

We reformulated the definition of an algebra via the commuting diagrams (A.0.26) because we want to *dualize* our objects and notions. The proceeding is now as follows: simply reverse arrows and demand objects to fill the gaps of the maps that do not fit any more in the diagrams.

Definition A.0.12 (Coalgebra) *We call $(\mathcal{C}, +, \Delta, \epsilon, \mathbb{k})$ a coalgebra over a field \mathbb{k} if there is a linear coproduct*

$$\Delta: \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C} \quad (\text{A.0.27})$$

which is coassociative and a linear counit

$$\epsilon: \mathcal{C} \rightarrow \mathbb{k}. \quad (\text{A.0.28})$$

The coassociativity axiom and the axioms a counit has to satisfy are denoted in the commutativity of the three diagrams of (A.0.29), respectively.

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} & \xleftarrow{1 \otimes \Delta} & \mathcal{C} \otimes \mathcal{C} \\ \Delta \otimes 1 \uparrow & & \uparrow \Delta \\ \mathcal{C} \otimes \mathcal{C} & \xleftarrow{\Delta} & \mathcal{C} \end{array} \quad \begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} & & \\ \epsilon \otimes 1 \downarrow & \swarrow \Delta & \\ \mathbb{k} \otimes \mathcal{C} = \mathcal{C} & & \end{array} \quad \begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} & & \\ 1 \otimes \epsilon \downarrow & \swarrow \Delta & \\ \mathcal{C} \otimes \mathbb{k} = \mathcal{C} & & \end{array} \quad (\text{A.0.29})$$

The axioms of a coalgebra depicted in the commutativity of the diagrams (A.0.29) read in formulas

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta \quad (\text{A.0.30})$$

and

$$(\epsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \epsilon) \circ \Delta, \quad (\text{A.0.31})$$

where $1: \mathcal{C} \rightarrow \mathcal{C}$ denotes the identity on \mathcal{C} .

Remark A.0.13 One could wonder what part of $\Delta(c) \in \mathcal{C} \otimes \mathcal{C}$ for a $c \in \mathcal{C}$ is applied to Δ, ϵ or 1 in Eq. (A.0.30) and Eq. (A.0.31), respectively. We want to discuss this problem. $\Delta(c) \in \mathcal{C} \otimes \mathcal{C}$ is a linear combination of elements $c_1 \otimes c_2 \in \mathcal{C} \otimes \mathcal{C}$ for $c_1, c_2 \in \mathcal{C}$, i.e. $\Delta(c) = \sum_{i=1}^n c_{1_i} \otimes c_{2_i}$ for a finite number $n \in \mathbb{N}$. Having this in mind we use the short notation (called **Sweedler's notation**)

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}, \quad (\text{A.0.32})$$

thus sometimes we even omit the sum. Then equation (A.0.30) reads

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}, \quad (\text{A.0.33})$$

where we used Eq. (A.0.32) again for the elements $\Delta(c_{(1)}) \in \mathcal{C} \otimes \mathcal{C}$ and $\Delta(c_{(2)}) \in \mathcal{C} \otimes \mathcal{C}$, respectively, while equation (A.0.31) reads

$$\epsilon(c_{(1)})c_{(2)} = c = \epsilon(c_{(2)})c_{(1)}, \quad (\text{A.0.34})$$

where we again used the isomorphisms stated in the proof of Lemma A.0.8. Motivated by equation (A.0.33) we moreover write

$$c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}. \quad (\text{A.0.35})$$

More general we define elements of the form $c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(r)} \in \mathcal{C} \otimes \cdots \otimes \mathcal{C}$ for any $r \in \mathbb{N}$ and $c \in \mathcal{C}$ by

$$\begin{aligned} c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(r)} &= (\Delta \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(r-1)\text{-times}}) \\ &\quad \circ (\Delta \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(r-2)\text{-times}}) \\ &\quad \vdots \\ &\quad \circ \Delta(c). \end{aligned}$$

For $r = 1$ and $r = 2$ we get back the definitions in Eq. (A.0.32) and Eq. (A.0.35), respectively. As in this two cases also permutations of Δ and 1 in the defining equations lead to $c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(r)}$ as a consequence of the coassociativity of Δ . Let us prove this for $r = 3$.

$$\begin{aligned} (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1) \circ \Delta(c) &= (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1)(c_{(1)} \otimes c_{(2)}) \\ &= (\Delta \otimes 1 \otimes 1)(c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}) \\ &= c_{(1)(1)(1)} \otimes c_{(1)(1)(2)} \otimes c_{(1)(2)} \otimes c_{(2)} \end{aligned}$$

$$\begin{aligned}
&= c_{(1)(1)} \otimes c_{(1)(2)(1)} \otimes c_{(1)(2)(2)} \otimes c_{(2)} \\
&= (1 \otimes \Delta \otimes 1) \circ (\Delta \otimes 1) \circ \Delta(c),
\end{aligned}$$

where we used the coassociativity in the fourth equation for the element $c_{(1)}$. Similarly for the other permutations. By induction this result is true for any $r \in \mathbb{N}$. As another short notation that we sometimes use, define

$$\Delta^{r-1}(c) = c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(r)}, \quad (\text{A.0.36})$$

where we set $\Delta^0 = \Delta$. While Δ adds another \mathcal{C} , the counit ϵ reduces its argument about one \mathcal{C} . More precise,

$$\begin{aligned}
(\epsilon \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r\text{-times}}) \circ \Delta^r(c) &= (1 \otimes \epsilon \otimes 1 \otimes \cdots \otimes 1) \circ \Delta^r(c) \\
&= \vdots \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{r\text{-times}} \otimes \epsilon) \circ \Delta^r(c) \\
&= c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(r)} \\
&= \Delta^{r-1}(c).
\end{aligned}$$

We prove this again only for $r = 3$.

$$\begin{aligned}
(\epsilon \otimes 1 \otimes 1 \otimes 1) \circ \Delta^3(c) &= (\epsilon \otimes 1 \otimes 1 \otimes 1)(c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} \otimes c_{(3)}) \\
&= \epsilon(c_{(1)(1)})c_{(1)(2)} \otimes c_{(2)} \otimes c_{(3)} \\
&= c_{(1)} \otimes c_{(2)} \otimes c_{(3)},
\end{aligned}$$

where we used the counit axiom for the element $c_{(1)}$ in the last equation. The other permutations work analogous.

Now after defining algebras and coalgebras we come to their morphisms.

Definition A.0.14 (Algebra and Coalgebra Map) A \mathbb{k} -linear map $f: \mathcal{A} \rightarrow \mathcal{B}$ between two algebras \mathcal{A} and \mathcal{B} over \mathbb{k} is said to be an algebra map if it satisfies for all $a, b \in \mathcal{A}$

$$f(ab) = f(a)f(b) \text{ and } f(1_{\mathcal{A}}) = 1_{\mathcal{B}}, \quad (\text{A.0.37})$$

where $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ denote the units of \mathcal{A} and \mathcal{B} , respectively. A \mathbb{k} -linear map $f: \mathcal{C} \rightarrow \mathcal{D}$ between two coalgebras \mathcal{C} and \mathcal{D} over \mathbb{k} is said to be a coalgebra map if it satisfies

$$(f \otimes f) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{D}} \circ f \text{ and } \epsilon_{\mathcal{D}} \circ f = \epsilon_{\mathcal{C}}. \quad (\text{A.0.38})$$

Remark A.0.15 One can structure the tensor product of two (co)algebras as a (co)algebra again. Explicitly, we define for two algebras \mathcal{A} and \mathcal{B} a product on the vector space $\mathcal{A} \otimes \mathcal{B}$ by

$$(a \otimes c) \cdot (b \otimes d) := (ab \otimes cd), \quad (\text{A.0.39})$$

for all $a, b \in \mathcal{A}$ and $c, d \in \mathcal{B}$ and extend this linearly to $\mathcal{A} \otimes \mathcal{B}$. Because of the associativity of the product on \mathcal{A} and \mathcal{B} , respectively, we get for $a, b, x \in \mathcal{A}$, $c, d, y \in \mathcal{B}$

$$((a \otimes c) \cdot (b \otimes d)) \cdot (x \otimes y) = (ab \otimes cd) \cdot (x \otimes y)$$

$$\begin{aligned}
&= (ab)x \otimes (cd)y \\
&= a(bx) \otimes c(dy) \\
&= (a \otimes c) \cdot ((b \otimes d) \cdot (x \otimes y)),
\end{aligned}$$

which is the associativity of \cdot . The unit of $\mathcal{A} \otimes \mathcal{B}$ is the tensor product of the units on \mathcal{A} and \mathcal{B} , i.e. $1_{\mathcal{A} \otimes \mathcal{B}} = 1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$ which can be seen via

$$1_{\mathcal{A} \otimes \mathcal{B}} \cdot (a \otimes b) = 1_{\mathcal{A}} a \otimes 1_{\mathcal{B}} b = a \otimes b = (a \otimes b) \cdot 1_{\mathcal{A} \otimes \mathcal{B}},$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. For two coalgebras \mathcal{C} and \mathcal{D} we define the coproduct on $\mathcal{C} \otimes \mathcal{D}$ for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ by

$$\Delta(c \otimes d) = (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}) \quad (\text{A.0.40})$$

and extend this linearly to $\mathcal{C} \otimes \mathcal{D}$. We used again the notation introduced in Eq. (A.0.32). The coassociativity is satisfied because for $c \in \mathcal{C}$ and $d \in \mathcal{D}$ one has

$$\begin{aligned}
(\Delta \otimes 1) \circ \Delta(c \otimes d) &= (\Delta \otimes 1)(c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}) \\
&= (c_{(1)(1)} \otimes d_{(1)(1)}) \otimes (c_{(1)(2)} \otimes d_{(1)(2)}) \otimes (c_{(2)} \otimes d_{(2)}) \\
&= (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)(1)} \otimes d_{(2)(1)}) \otimes (c_{(2)(2)} \otimes d_{(2)(2)}) \\
&= (1 \otimes \Delta) \circ \Delta(c \otimes d)
\end{aligned}$$

by the coassociativity in \mathcal{C} and \mathcal{D} , respectively. The 1 in the equation above denotes the identity map on $\mathcal{C} \otimes \mathcal{D}$. The counit of $\mathcal{C} \otimes \mathcal{D}$ is the tensor product of the counits of \mathcal{C} and \mathcal{D} , i.e. $\epsilon = \epsilon_{\mathcal{C}} \otimes \epsilon_{\mathcal{D}}$, since for $c \in \mathcal{C}$, $d \in \mathcal{D}$ one has

$$\begin{aligned}
(\epsilon \otimes 1) \circ \Delta(c \otimes d) &= (\epsilon \otimes 1)(c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}) \\
&= (\epsilon_{\mathcal{C}}(c_{(1)}) \otimes \epsilon_{\mathcal{D}}(d_{(1)})) \otimes (c_{(2)} \otimes d_{(2)}) \\
&= c \otimes d,
\end{aligned}$$

where the last equation follows from the counit property on \mathcal{C} and \mathcal{D} , respectively. The other counit axiom follows similarly.

The following should give an argument why one calls the process of passing from an algebra to a coalgebra (and back) *dualizing*.

Proposition A.0.16 *Consider a coalgebra \mathcal{C} and the adjoint maps*

$$\cdot: \mathcal{C}^* \otimes \mathcal{C}^* \rightarrow \mathcal{C}^* \quad (\text{A.0.41})$$

and

$$\eta: \mathbb{k} \rightarrow \mathcal{C}^* \quad (\text{A.0.42})$$

of the coproduct and counit, respectively. More explicit, one has for $\phi, \psi \in \mathcal{C}^*$, $c \in \mathcal{C}$ and $\lambda \in \mathbb{k}$

$$(\phi \cdot \psi)(c) = (\phi \otimes \psi) \circ \Delta(c) \quad (\text{A.0.43})$$

and

$$\eta(\lambda)(c) = \lambda \epsilon(c). \quad (\text{A.0.44})$$

The product \cdot and the unit η make the dual space \mathcal{C}^* into an algebra. The unit element is ϵ . Conversely, if \mathcal{A} is a finite-dimensional algebra, \mathcal{A}^* gets a coalgebra structure in the same fashion.

PROOF: If \mathcal{C} is a coalgebra Eq. (A.0.43) defines an associative product on \mathcal{C}^* because for $\phi, \psi, \chi \in \mathcal{C}^*$ and $c \in \mathcal{C}$ one has

$$\begin{aligned} ((\phi \cdot \psi) \cdot \chi)(c) &= (\phi \cdot \psi)(c_{(1)})\chi(c_{(2)}) \\ &= \phi(c_{(1)(1)})\psi(c_{(1)(2)})\chi(c_{(2)}) \\ &= (\phi \otimes \psi \otimes \chi) \circ (\Delta \otimes 1) \circ \Delta(c) \\ &= (\phi \otimes \psi \otimes \chi) \circ (1 \otimes \Delta) \circ \Delta(c) \\ &= (\phi \cdot (\psi \cdot \chi))(c), \end{aligned}$$

where we used the coassociativity of Δ on \mathcal{C} in the fourth equation. The unit axiom of the map defined in Eq. (A.0.44) also follows for $\lambda \in \mathbb{k}$, $\phi \in \mathcal{C}^*$:

$$\begin{aligned} \cdot((\eta \otimes 1)(\lambda \otimes \phi))(c) &= \cdot(\eta(\lambda) \otimes \phi)(c) \\ &= \lambda \epsilon(c_{(1)})\phi(c_{(2)}) \\ &= (\lambda \epsilon \otimes \phi) \circ \Delta(c) \\ &= (\lambda 1_{\mathbb{k}} \otimes \phi) \circ (\epsilon \otimes 1) \circ \Delta(c) \\ &= \lambda \phi(c), \end{aligned}$$

where the last equation follows from the counit property of ϵ on \mathcal{C} . The other unit axiom follows similarly. Conversely, if \mathcal{A} is an algebra we can define a coproduct and a counit via Eq. (A.0.43) and Eq. (A.0.44) where one has to read the equations from right to left. This is possible since in the finite-dimensional situation we can identify \mathcal{A} with the dual space of \mathcal{A}^* . Thus $(a \otimes b) \circ \Delta(\phi) = (a \cdot b)(\phi)$ and $\lambda \epsilon(\phi) = \eta(\lambda)(\phi)$ for $a, b \in \mathcal{A} = \mathcal{A}^{**}$, $\lambda \in \mathbb{k}$ and $\phi \in \mathcal{A}^*$. Then the same equations above reordered give the coassociativity of Δ on \mathcal{A}^* by the associativity of \cdot on \mathcal{A} and the counit axiom of ϵ on \mathcal{A}^* from the unit axiom of η on \mathcal{A} . This concludes the proof. \square

The situation gets interesting if we combine both, the features of an algebra and a coalgebra, in one space. Moreover, the algebra and coalgebra structure should be compatible. This leads to the next

Definition A.0.17 (Bialgebra) We call $(H, +, \cdot, \eta, \Delta, \epsilon, \mathbb{k})$ a bialgebra over a field \mathbb{k} if $(H, +, \mathbb{k})$ is an algebra and a coalgebra over \mathbb{k} such that the respective structures are compatible, i.e. $\Delta: H \rightarrow H \otimes H$ and $\epsilon: H \rightarrow \mathbb{k}$ are algebra maps. Explicitly, this compatibility reads for $g, h \in H$

$$\Delta(gh) = \Delta(g)\Delta(h), \quad (\text{A.0.45})$$

$$\Delta(1_H) = 1_H \otimes 1_H, \quad (\text{A.0.46})$$

$$\epsilon(gh) = \epsilon(g)\epsilon(h), \quad (\text{A.0.47})$$

$$\epsilon(1_H) = 1_{\mathbb{k}}. \quad (\text{A.0.48})$$

Bialgebras are self dual objects. To illustrate this we prove a characterization of bialgebras via their algebra structure. Remark that bialgebras are defined in terms of their coalgebra structure.

Proposition A.0.18 An algebra and coalgebra $(H, +, \cdot, \eta, \Delta, \epsilon, \mathbb{k})$ is a bialgebra if and only if $\cdot: H \otimes H \rightarrow H$ and $\eta: \mathbb{k} \rightarrow H$ are coalgebra maps, i.e.

$$(\cdot \otimes \cdot) \circ \Delta_{H \otimes H} = \Delta_H \circ \cdot, \quad (\text{A.0.49})$$

$$\epsilon_H \circ \cdot = \epsilon_{H \otimes H}, \quad (\text{A.0.50})$$

$$(\eta \otimes \eta) \circ \Delta_{\mathbb{k}} = \Delta_H \circ \eta, \quad (\text{A.0.51})$$

$$\epsilon_H \circ \eta = \epsilon_{\mathbb{k}}. \quad (\text{A.0.52})$$

PROOF: Assume that $(H, +, \cdot, \eta, \Delta, \epsilon, \mathbb{k})$ is a bialgebra. It is just a matter of calculation to verify Eq. (A.0.49) and the equations following. Thus let $g, h \in H$ and $\lambda \in \mathbb{k}$.

$$\begin{aligned}
(\cdot \otimes \cdot) \circ \Delta_{H \otimes H}(g \otimes h) &= (\cdot \otimes \cdot)(g_{(1)} \otimes h_{(1)} \otimes g_{(2)} \otimes h_{(2)}) \\
&= (g_{(1)} \cdot h_{(1)}) \otimes (g_{(2)} \cdot h_{(2)}) \\
&= (g_{(1)} \otimes g_{(2)}) \cdot (h_{(1)} \otimes h_{(2)}) \\
&= \Delta_H(g) \cdot \Delta_H(h) \\
&= \Delta_H \circ \cdot (g \otimes h),
\end{aligned}$$

where \cdot in the third line denotes the product on the tensor product $H \otimes H$ defined in Eq. (A.0.39).

$$\begin{aligned}
\epsilon_H \circ \cdot (g \otimes h) &= \epsilon_H(g \cdot h) \\
&= \epsilon_H(g) \cdot \epsilon_H(h) \\
&= \epsilon_H(g) \otimes \epsilon_H(h) \\
&= \epsilon_{H \otimes H}(g \otimes h),
\end{aligned}$$

where the second equation holds since ϵ_H is an algebra map, the third equation is due to one isomorphism of Lemma A.0.8 and the last equation is the definition of $\epsilon_{H \otimes H}$.

$$\begin{aligned}
(\eta \otimes \eta) \circ \Delta_{\mathbb{k}}(\lambda) &= (\eta \otimes \eta)(\lambda_{(1)} \otimes \lambda_{(2)}) \\
&= \eta(\lambda_{(1)}) \otimes \eta(\lambda_{(2)}) \\
&= (\lambda_{(1)} 1_H) \otimes (\lambda_{(2)} 1_H) \\
&= \lambda_{(1)} \lambda_{(2)} 1_H \otimes 1_H \\
&= \lambda_{(1)} \lambda_{(2)} \Delta_H(1_H) \\
&= \Delta_H((\lambda_{(1)} \otimes \lambda_{(2)}) 1_H) \\
&= \Delta_H(\eta(\lambda)),
\end{aligned}$$

where the equation next to the last follows from the fact that Δ_H is \mathbb{k} -linear and again by an isomorphism of Lemma A.0.8. Finally,

$$\epsilon_H \circ \eta(\lambda) = \epsilon_H(\lambda 1_H) = \lambda \epsilon_H(1_H) = \lambda 1_{\mathbb{k}} = \epsilon_{\mathbb{k}}(\lambda),$$

where the second equation follows because ϵ_H is \mathbb{k} -linear, the third equation holds because ϵ_H is an algebra map and the last equation is the definition of $\epsilon_{\mathbb{k}}$. Conversely, assume that equation (A.0.49) and the three equations following hold. The same equations of the first part of this proof reordered prove that Δ and ϵ are algebra maps. Explicitly, we calculate for $g, h \in H$

$$\begin{aligned}
\Delta_H(g \cdot h) &= \Delta_H \circ \cdot (g \otimes h) \\
&= (\cdot \otimes \cdot) \circ \Delta_{H \otimes H}(g \otimes h) \\
&= (\cdot \otimes \cdot)(g_{(1)} \otimes h_{(1)} \otimes g_{(2)} \otimes h_{(2)}) \\
&= (g_{(1)} \cdot h_{(1)}) \otimes (g_{(2)} \cdot h_{(2)}) \\
&= (g_{(1)} \otimes g_{(2)}) \cdot (h_{(1)} \otimes h_{(2)}) \\
&= \Delta_H(g) \cdot \Delta_H(h),
\end{aligned}$$

where the second equation holds since \cdot is a coalgebra map and we used again the multiplication on the tensor product $H \otimes H$.

$$1_H \otimes 1_H = \eta(1_{\mathbb{k}}) \otimes \eta(1_{\mathbb{k}}) = (\eta \otimes \eta) \circ \Delta_{\mathbb{k}}(1_{\mathbb{k}}) = \Delta_H(\eta(1_{\mathbb{k}})) = \Delta_H(1_H),$$

where we used that $\Delta_{\mathbb{k}}(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes 1_{\mathbb{k}}$ and that η is a coalgebra map.

$$\epsilon_H(g \cdot h) = \epsilon_H \circ \cdot (g \otimes h) = \epsilon_{H \otimes H}(g \otimes h) = \epsilon_H(g) \otimes \epsilon_H(h) = \epsilon_H(g) \cdot \epsilon_H(h),$$

where we used that \cdot is a coalgebra map, the equation next to the last equation is the definition of $\epsilon_{H \otimes H}$ and the last equation follows from Lemma A.0.8. Remark that condition $\epsilon(1_H) = 1_{\mathbb{k}}$ is always satisfied since \mathbb{k} is a field. This concludes the proof. \square

Now we can introduce the definition of a *Hopf algebra*. Essentially, a Hopf algebra is a bialgebra over a field (or sometimes even over a commutative ring) with one extra structure.

Definition A.0.19 (Hopf Algebra) *We call $(H, +, \cdot, \eta, \Delta, \epsilon, S, \mathbb{k})$ a Hopf algebra over a field \mathbb{k} if it is a bialgebra over \mathbb{k} and there is a \mathbb{k} -linear map*

$$S: H \rightarrow H \tag{A.0.53}$$

called an antipode of H satisfying

$$\cdot(S \otimes 1) \circ \Delta = \eta \circ \epsilon = \cdot(1 \otimes S) \circ \Delta. \tag{A.0.54}$$

In other words, an algebra and coalgebra H is a Hopf algebra if and only if the diagrams of (A.0.55) and (A.0.56) commute, where (A.0.55) encodes the compatibility axioms of a bialgebra and (A.0.56) the antipode axiom.

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{\cdot} & H & \xrightarrow{\Delta} & H \otimes H & & H & \xrightarrow{\epsilon} & \mathbb{k} & & H & \xleftarrow{\eta} & \mathbb{k} \\ \Delta \otimes \Delta \downarrow & & & & \uparrow \cdot \otimes \cdot & & \uparrow \cdot & \nearrow \epsilon \otimes \epsilon & & & \downarrow \Delta & \nearrow \eta \otimes \eta & \\ H \otimes H \otimes H \otimes H & \xrightarrow{1 \otimes \sigma \otimes 1} & H \otimes H \otimes H \otimes H & & H \otimes H & & H \otimes H & & & & H \otimes H & & \end{array} \tag{A.0.55}$$

Here $\sigma: H \otimes H \ni g \otimes h \mapsto h \otimes g \in H \otimes H$ denotes the tensor product twist.

$$\begin{array}{ccccc} H & \xrightarrow{\epsilon} & \mathbb{k} & \xrightarrow{\eta} & H \\ \Delta \downarrow & & & & \uparrow \cdot \\ H \otimes H & \xrightarrow{1 \otimes S, S \otimes 1} & H \otimes H & & \end{array} \tag{A.0.56}$$

The next proposition gives some properties of the antipode of a Hopf algebra. On the one hand they are useful if one wants to calculate examples on the other hand they are useful to clarify if a map is the antipode of a Hopf algebra.

Proposition A.0.20 *The antipode S of a Hopf algebra H is unique. Moreover, S is an antialgebra map, i.e. for all $g, h \in H$ we have*

$$S(g \cdot h) = S(h) \cdot S(g) \tag{A.0.57}$$

$$S(1) = 1 \tag{A.0.58}$$

and S is an anticoalgebra map, i.e. for all $g \in H$ one has

$$(S \otimes S) \circ \Delta(g) = \sigma \circ \Delta \circ S(g) \tag{A.0.59}$$

$$\epsilon \circ S(g) = \epsilon(g). \tag{A.0.60}$$

PROOF: We follow [70, Proposition 1.3.1]. Assume there are two antipodes S and S' of a bialgebra H . We first need a useful identity, namely we calculate for $g \in H$

$$g_{(1)}S(g_{(2)}) = (\cdot(1 \otimes S) \circ \Delta)(g) = \eta \circ \epsilon(g) = \epsilon(g)1_H,$$

where we used the antipode axiom. Thus

$$g_{(1)}S(g_{(2)}) = \epsilon(g)1_H \quad (\text{A.0.61})$$

and in the same fashion

$$S(g_{(1)})g_{(2)} = \epsilon(g)1_H \quad (\text{A.0.62})$$

if we use the second antipode with 1 and S changed. These equations are remarkable because they show that $S(g_{(1)})$ is somehow an inverse to $g_{(2)}$ (or $g_{(1)}$ and $g_{(2)}$ exchanged) and equations (A.0.61) and (A.0.62) remind in a way of cancelling gg^{-1} and $g^{-1}g$ to the unit element of a group. Consider the proof of Proposition 1.3.1 in [70] for more details to this idea. Now it is just a matter of calculation. Let $g \in H$, then

$$\begin{aligned} S'(g) &= S'(g_{(1)})\epsilon(g_{(2)}) \\ &= S'(g_{(1)})g_{(2)(1)}S(g_{(2)(2)}) \\ &= S'(g_{(1)(1)})g_{(1)(2)}S(g_{(2)}) \\ &= (\cdot(S' \otimes 1) \circ \Delta(g_{(1)}))S(g_{(2)}) \\ &= (\eta \circ \epsilon(g_{(1)}))S(g_{(2)}) \\ &= \epsilon(g_{(1)})S(g_{(2)}) \\ &= S(g) \end{aligned}$$

This proves $S' = S$ what implies that the antipode of a Hopf algebra is unique. The next step is to prove that S is an antialgebra map. It is easy to get

$$S(1_H) = 1_H S(1_H) = (1_H)_{(1)}S((1_H)_{(2)}) = \epsilon(1_H)1_H = 1_H.$$

This is again just Eq. (A.0.61) with $\Delta(1_H) = (1_H)_{(1)} \otimes (1_H)_{(2)} = 1_H \otimes 1_H$ what is true since Δ is an algebra map and in the last equation we used $\epsilon(1_H) = 1_H$ what is true since ϵ is an algebra map. The other property is proved step by step. First of all, the antipode axiom for an element gh with $g, h \in H$ reads

$$S(g_{(1)}h_{(1)})g_{(2)}h_{(2)} = \epsilon(gh)1_H = \epsilon(g)\epsilon(h)1_H, \quad (\text{A.0.63})$$

since ϵ is an algebra map. If we replace h with $h_{(1)}$, where of course $\Delta(h) = h_{(1)} \otimes h_{(2)}$ and add $h_{(2)}$ as a tensor factor in equation (A.0.63) we obtain

$$S(g_{(1)}h_{(1)(1)})g_{(2)}h_{(1)(2)} \otimes h_{(2)} = \epsilon(g)\epsilon(h_{(1)})1_H \otimes h_{(2)} = \epsilon(g)1_H \otimes h, \quad (\text{A.0.64})$$

where we used $(\epsilon \otimes 1) \circ \Delta(h) = h$ which holds by the counit axiom. If we apply S to the second tensor factor of (A.0.64) multiply both tensor factors via \cdot , Eq. (A.0.64) reads

$$S(g_{(1)}h_{(1)})g_{(2)}h_{(2)}S(h_{(3)}) = \epsilon(g)S(h) \quad (\text{A.0.65})$$

by the associativity of \cdot . Together with Eq. (A.0.61) and the counit axiom we see that Eq. (A.0.65) transforms to

$$\epsilon(g)S(h) = S(g_{(1)}h_{(1)})g_{(2)}h_{(2)}S(h_{(3)})$$

$$\begin{aligned}
&= S(g_{(1)}h_{(1)})g_{(2)}\epsilon(h_{(2)}) \\
&= S(g_{(1)}h)g_{(2)}
\end{aligned}$$

If we replace g by $g_{(1)}$ and add $g_{(2)}$ on both sides as a tensor factor the last equation is equivalent to

$$S(g_{(1)(2)}h)g_{(1)(2)} \otimes g_{(2)} = \epsilon(g_{(1)})S(h) \otimes g_{(2)} = S(h) \otimes g, \quad (\text{A.0.66})$$

where we used the counit axiom. Finally, we apply S to the second tensor factor of Eq. (A.0.66) and multiply on both sides of Eq. (A.0.66) via \cdot and obtain by the associativity of \cdot , Eq. (A.0.61) and the counit axiom

$$S(h)S(g) = S(g_{(1)}h)h_{(2)(1)}S(h_{(2)(2)}) = S(g_{(1)}h)\epsilon(g_{(2)}) = S(gh).$$

Thus S is an antialgebra map. The anticoalgebra map properties are a bit easier to check. First of all, for $g \in H$ one has

$$\begin{aligned}
\epsilon(S(g)) &= \epsilon(S(g_{(1)}))\epsilon(g_{(2)}) = \epsilon(S(g_{(1)})g_{(2)}) \\
&= \epsilon(\epsilon(g)1_H) = \epsilon(g)\epsilon(1_H) \\
&= \epsilon(h),
\end{aligned}$$

where we used the linearity of ϵ together with Eq. (A.0.62) and that ϵ is an algebra map. Finally, we calculate for $g \in H$

$$\begin{aligned}
\sigma \circ (S \otimes S) \circ \Delta(g) &= S(g_{(2)}) \otimes S(g_{(1)}) \\
&= (S(g_{(1)}))_{(1)}g_{(2)(1)}S(g_{(4)}) \otimes (S(g_{(1)}))_{(2)}g_{(2)(2)}S(g_{(3)}) \\
&= (S(g_{(1)}))_{(1)}g_{(2)}S(g_{(3)}) \otimes (S(g_{(1)}))_{(2)} \\
&= S(g)_{(1)} \otimes S(g)_{(1)} \\
&= \Delta \circ S(g),
\end{aligned}$$

using the same techniques as above. This implies $(S \otimes S) \circ \Delta = \sigma \circ \Delta \circ S$. \square

Remark that if H and G are Hopf algebras with antipodes S and S' respectively, also the tensor product $H \otimes G$ is a Hopf algebra. The bialgebra structure is given by the usual tensor bialgebra structure, while the unique antipode is $S \otimes S'$. We did not yet define the corresponding morphisms:

Definition A.0.21 (Hopf Algebra Map) *A map $f: H \rightarrow G$ between two Hopf algebras H and G which is an algebra map and a coalgebra map is said to be a Hopf algebra map if it satisfies*

$$S' \circ f = f \circ S, \quad (\text{A.0.67})$$

where S and S' are the antipodes of H and G , respectively.

Appendix B

Semisimple Lie Algebras and Iwasawa Decomposition

This appendix treats several topics in Lie theory. We develop the notion of reductive Lie algebras and stress that the difficulty in their classification shifts to the classification of simple Lie algebras. We do not prove the classification, but provide some notions and techniques that are interesting for their own. The radical of the Killing form controls the abelian ideals of the Lie algebra and is an indicator for semisimple Lie algebras. Then the eigenspaces of a Cartan involution give a first useful decomposition of a semisimple Lie algebra: the Cartan decomposition. By considering root systems we can even refine this description and obtain a Iwasawa decomposition. All these decompositions can also be done on Lie group level.

B.1 The Classification of Reductive Lie Algebras

Here we summarize some very basic facts about Lie algebras. It is not about giving a full description of the theory, but to come to the definition of a semisimple Lie algebra as fast as possible. For an introduction to Lie algebras consider e.g. [49] or [55]. We follow [57, Chapter 2]. Let \mathbb{k} be a field.

Definition B.1.1 (Lie Algebra) *A \mathbb{k} -vector space \mathfrak{g} endowed with a \mathbb{k} -bilinear map*

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \tag{B.1.1}$$

*which satisfies $[x, x] = 0$ for all $x \in \mathfrak{g}$ and the **Jacobi identity***

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0, \tag{B.1.2}$$

*for all $x, y, z \in \mathfrak{g}$, is said to be a Lie algebra over \mathbb{k} . The map defined in Eq. (B.1.1) is said to be the **Lie bracket** of \mathfrak{g} and one often writes $(\mathfrak{g}, [\cdot, \cdot])$ to denote the Lie algebra together with its bracket.*

By using the short notation

$$[\mathfrak{h}, \mathfrak{l}] = \text{span}_{\mathbb{k}}\{[h, \ell] \mid h \in \mathfrak{h}, \ell \in \mathfrak{l}\}, \tag{B.1.3}$$

where \mathfrak{h} and \mathfrak{l} are arbitrary subsets of a Lie algebra \mathfrak{g} over \mathbb{k} , one can define some substructures and important types of Lie algebras:

Definition B.1.2 Consider a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ over \mathbb{k} .

- i.) A subset \mathfrak{h} of \mathfrak{g} is said to be
 - a.) a **Lie subalgebra** of \mathfrak{g} if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.
 - b.) an **ideal** of \mathfrak{g} if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.
 - c.) a **proper ideal** of \mathfrak{g} if it is an ideal of \mathfrak{g} but not equal to $\{0\}$ or \mathfrak{g} .
- ii.) \mathfrak{g} is said to be
 - a.) **abelian** if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$.
 - b.) **simple** if \mathfrak{g} is not abelian and there is no proper ideal contained in \mathfrak{g} .
 - c.) **semisimple** if \mathfrak{g} can be written as a direct sum of simple Lie algebras.
 - d.) **reductive** if \mathfrak{g} can be written as a direct sum of simple and abelian Lie algebras.

It is clear from the definition that every proper ideal is an ideal and every ideal is a Lie subalgebra. $\{0\}$ and \mathfrak{g} are generic ideals of \mathfrak{g} . Moreover, every abelian or simple or semisimple Lie algebra is reductive and every simple Lie algebra is semisimple. The Lie groups $\mathrm{SO}(3)$, $\mathrm{SL}(3, \mathbb{R})$ and $\mathcal{L}_3^{\uparrow, +}$ we consider in Section 2.4 are semisimple (c.f. [90, Section 0.8.3]).

Remark B.1.3 Let $n \in \mathbb{N}$. One can prove that any n -dimensional abelian Lie algebra is isomorphic to

$$\bigoplus_{k=1}^n \mathfrak{u}(1), \quad (\text{B.1.4})$$

where $\mathfrak{u}(1)$ is a 1-dimensional vector space endowed with the Lie bracket that is identically zero. A more difficult approach is needed to classify reductive and for this semisimple and simple Lie algebras. This is called the **Cartan-Killing Classification**. For a proof that uses Dynkin diagrams consider [20].

B.2 Semisimple Lie Algebras

We start by defining a very central element in the theory of semisimple Lie algebras and follow [87], but also refer to [58, Chapter 2]. The eigenspaces of a Cartan involution of a Lie algebra provide a useful decomposition. For the definition we need the Killing form of a Lie algebra. Later we see that the non-degeneracy of this object characterizes complex semisimple Lie algebras. For a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and an element $x \in \mathfrak{g}$ we denote the *adjoint endomorphism* on \mathfrak{g} corresponding to x by

$$\mathrm{ad}(x): \mathfrak{g} \ni y \mapsto [x, y] \in \mathfrak{g}. \quad (\text{B.2.1})$$

Definition B.2.1 (Killing Form) Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over a field \mathbb{k} . The map

$$\kappa: \mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto \mathrm{tr}(\mathrm{ad}(x) \mathrm{ad}(y)) \in \mathbb{k} \quad (\text{B.2.2})$$

is said to be the *Killing form* on \mathfrak{g} .

Already for an arbitrary Lie algebra κ has interesting properties:

Proposition B.2.2 Consider a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ over a field \mathbb{k} . Then the following statements hold.

i.) The map κ defined in Eq. (B.2.2) is a well-defined symmetric bilinear form.

ii.) κ is **associative**, i.e. for all $x, y, z \in \mathfrak{g}$ the equation

$$\kappa([x, y], z) = \kappa(x, [y, z]) \quad (\text{B.2.3})$$

holds.

iii.) κ is invariant under any automorphism ρ of \mathfrak{g} , i.e. for all $x, y \in \mathfrak{g}$ one has

$$\kappa(\rho(x), \rho(y)) = \kappa(x, y). \quad (\text{B.2.4})$$

iv.) The **radical**

$$S = \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \text{ for all } y \in \mathfrak{g}\} \quad (\text{B.2.5})$$

is an ideal of \mathfrak{g} .

PROOF: Since $\text{ad}(x)\text{ad}(y)$ is an endomorphism which is mapped to tr by the trace for all $x, y \in \mathfrak{g}$ the map κ is well-defined. By the linearity of the trace and of the map

$$\text{ad}: \mathfrak{g} \ni x \mapsto \text{ad}(x) \in \text{End}(\mathfrak{g}) \quad (\text{B.2.6})$$

the bilinearity of κ is clear. The symmetry of κ is also clear by the cyclic permutation property of the trace. Now the second part is an easy calculation. For $x, y, z \in \mathfrak{g}$ one has

$$\begin{aligned} \kappa([x, y], z) &= \text{tr}(\text{ad}([x, y])\text{ad}(z)) = \text{tr}([\text{ad}(x), \text{ad}(y)]\text{ad}(z)) \\ &= \text{tr}(\text{ad}(x)\text{ad}(y)\text{ad}(z) - \text{ad}(y)\text{ad}(x)\text{ad}(z)) \\ &= \text{tr}(\text{ad}(x)\text{ad}(y)\text{ad}(z) - \text{ad}(x)\text{ad}(z)\text{ad}(y)) \\ &= \text{tr}(\text{ad}(x)[\text{ad}(y), \text{ad}(z)]) = \text{tr}(\text{ad}(z)\text{ad}([y, z])) \\ &= \kappa(x, [y, z]), \end{aligned}$$

where we used again cyclic permutations of tr and two times the formula $\text{ad}([x, y]) = [\text{ad}(x), \text{ad}(y)]$ which holds because ad is a Lie algebra homomorphism. For the third statement take $x, y, z \in \mathfrak{g}$ such that $z = \rho(y)$. Since ρ is invertible, the equation $\rho([x, y]) = [\rho(x), \rho(y)]$ is equivalent to

$$\text{ad}(\rho(x))(z) = (\rho \circ \text{ad}(x) \circ \rho^{-1})(z). \quad (\text{B.2.7})$$

This implies

$$\kappa(\rho(x), \rho(y)) = \text{tr}(\text{ad}(\rho(x))\text{ad}(\rho(y))) = \text{tr}(\rho \circ (\text{ad}(x)\text{ad}(y)) \circ \rho^{-1}) = \kappa(x, y).$$

The last statement is a consequence of ii.). To prove this, take arbitrary elements $x \in S$ and $y \in \mathfrak{g}$. Then $[x, y] \in S$ because

$$\kappa([x, y], z) = \kappa(x, \underbrace{[y, z]}_{\in \mathfrak{g}}) = 0$$

for all $z \in \mathfrak{g}$ which proves the claim. \square

In particular, the last statement of this proposition is interesting for semisimple Lie algebras since there is always an ideal S in \mathfrak{g} , namely the radical of κ . If $S = \mathfrak{g}$ the Lie algebra would be rather boring, thus one traceable claim to hope for a semisimple Lie algebra should be $S = \{0\}$. Since κ is symmetric according to i.) this is equivalent to the postulate that κ is non-degenerate in \mathfrak{g} . Our aim is to prove that this is not only a necessary but also a sufficient condition to characterize complex semisimple Lie algebras. We first prove that S controls all non-proper abelian ideals.

Lemma B.2.3 *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over a field \mathbb{k} and J an abelian ideal of \mathfrak{g} . Then $J \subseteq S$.*

PROOF: Let $x \in J$ and $y \in \mathfrak{g}$ be arbitrary. Our first step is to show that the endomorphism

$$\text{ad}(x)\text{ad}(y) \tag{B.2.8}$$

is nilpotent. More explicit, we prove $(\text{ad}(x)\text{ad}(y))^2 = 0$. First of all $(\text{ad}(x)\text{ad}(y))(z) \in J$ for all $z \in \mathfrak{g}$ since

$$(\text{ad}(x)\text{ad}(y))(z) = [x, \underbrace{[y, z]}_{\in \mathfrak{g}}]. \tag{B.2.9}$$

Similarly we get for $z \in J$

$$(\text{ad}(x)\text{ad}(y))(z) \subseteq \text{ad}(x)(J) = \{0\}, \tag{B.2.10}$$

where the last equation follows since J is abelian and $x \in J$. Equation (B.2.10) combined with equation (B.2.9) gives $(\text{ad}(x)\text{ad}(y))^2 = 0$. Thus (B.2.8) is indeed nilpotent and for this traceless. We conclude

$$\kappa(x, y) = \text{tr}(\text{ad}(x)\text{ad}(y)) = 0$$

which shows $J \subseteq S$. □

For the proof of the next theorem we need another characterization of semisimple Lie algebras taken from [10, Theorem 6.5]: a Lie algebra is semisimple if and only if it has no non-zero abelian ideal.

Theorem B.2.4 *Consider a finite-dimensional complex Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. It is semisimple if and only if the Killing form κ is non-degenerate on \mathfrak{g} .*

PROOF: As argued before we prove that \mathfrak{g} has a non-zero abelian ideal if and only if $S \neq \{0\}$. If \mathfrak{g} has a non-zero abelian ideal J then $\{0\} \neq J \subseteq S$ according to Lemma B.2.3. Assume conversely $S \neq \{0\}$. Then **Cartan's Criterion** (c.f. [1, Corollary 3.3.14]) implies that S is a proper ideal of \mathfrak{g} since $\kappa(x, y) = 0$ for all $x, y \in S$. Thus \mathfrak{g} is not semisimple. □

In the following we often view a semisimple Lie algebra together with its Killing form. An object which is quite related to the Killing form is a Cartan involution.

Definition B.2.5 (Cartan Involution) *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a semisimple Lie algebra over a field \mathbb{k} . If there is an involution Θ on \mathfrak{g} , i.e. a automorphism of \mathfrak{g} that satisfies $\Theta^2 = 1$, such that*

$$B_{\Theta}(x, y) = -\kappa(x, \Theta(y)) \tag{B.2.11}$$

is a positive definite bilinear form, where $x, y \in \mathfrak{g}$, then Θ is said to be a Cartan involution on \mathfrak{g} .

One could ask in which situation such a Cartan involution exists and when it is unique. There is an answer for real semisimple Lie algebras:

Theorem B.2.6 *There is a Cartan involution on a real semisimple Lie algebra which is unique up to inner automorphisms.*

A proof can be found in [58, Theorem 3]. It uses Theorem B.2.4 and the existence of a compact real form of a finite-dimensional semisimple Lie algebra. We want to say some words to the last point. A real Lie algebra \mathfrak{g}_0 is said to be a *real form* of a complex Lie algebra \mathfrak{g} if one has

$$\mathfrak{g} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0. \quad (\text{B.2.12})$$

Conversely, a complex Lie algebra \mathfrak{g} is said to be a *complexification* of a real Lie algebra \mathfrak{g}_0 if Eq. (B.2.12) holds. A famous theorem of Cartan says that if \mathfrak{g} is a complex semisimple Lie algebra there is always a real form \mathfrak{g}_0 of \mathfrak{g} and \mathfrak{g}_0 is a compact Lie algebra (c.f. [89]).

Now consider a real semisimple Lie algebra \mathfrak{g} with a Cartan involution Θ . Since it is an involution Θ has exactly two eigenvalues ± 1 . The corresponding eigenspaces are commonly denoted by \mathfrak{k} and \mathfrak{p} .

Definition B.2.7 (Cartan Decomposition) *The decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad (\text{B.2.13})$$

of a real semisimple Lie algebra \mathfrak{g} is called the Cartan decomposition of \mathfrak{g} corresponding to Θ .

Proposition B.2.8 *The eigenspaces \mathfrak{k} and \mathfrak{p} of the Cartan decomposition (B.2.13) satisfy $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.*

PROOF: This is clear since Θ is an automorphism. For example for $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$ one has

$$\Theta([k, p]) = [\Theta(k), \Theta(p)] = [k, -p] = -[k, p]. \quad \square$$

B.3 Root Space Decomposition and Iwasawa Decomposition

To get in touch with Iwasawa decompositions we use roots and Cartan subalgebras and follow [57, Chapter 3]. There are many equivalent definitions of Cartan subalgebras, but we characterize them via semisimple elements. Recall that for an element x of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ the endomorphism $\text{ad}(x)$ is diagonalizable if there is a basis $\{x_1, x_2, \dots\}$ of \mathfrak{g} such that $[x, x_k]$ is proportional to x_k for every $k = 1, 2, \dots$. Moreover, an abelian subalgebra \mathfrak{h} of \mathfrak{g} is said to be *maximal abelian* if there is no abelian subalgebra \mathfrak{h}' of \mathfrak{g} such that $\mathfrak{h} \subsetneq \mathfrak{h}' \subsetneq \mathfrak{g}$.

Definition B.3.1 (Cartan Subalgebra) *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra.*

- i.) An element $x \in \mathfrak{g}$ is said to be semisimple if $\text{ad}(x)$ is diagonalizable.*
- ii.) If \mathfrak{g} is semisimple, a maximal abelian subalgebra \mathfrak{h} of \mathfrak{g} is said to be a Cartan subalgebra of \mathfrak{g} if it consists of semisimple elements.*

Can one guarantee the existence of a Cartan subalgebra? This is discussed in the next

Remark B.3.2 There are semisimple elements if the field \mathbb{k} corresponding to the semisimple Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is algebraically closed. This is clear because the condition of $\text{ad}(x)$ being diagonalizable for an $x \in \mathfrak{g}$ reduces to solutions $\lambda \in \mathbb{k}$ of the equation

$$\det(M - \lambda \mathbb{1}) = 0, \quad (\text{B.3.1})$$

where the entries of M are the structure constants M_{ij} which satisfy $[x, x_i] = \sum_j M_{ij}x_j$. Thus it is possible to choose a maximal set of linearly independent semisimple elements $H_i \in \mathfrak{g}$ that satisfy

$$[H_i, H_j] = 0. \quad (\text{B.3.2})$$

The subset spanned by these elements is a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . In general there are many Cartan subalgebras of \mathfrak{g} , but one can show that they have all the same dimension, which is said to be the **rank** of \mathfrak{g} .

Consider again a semisimple Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ over an algebraically closed field \mathbb{k} and choose a Cartan subalgebra \mathfrak{h} with basis $\{H_i\}$ as it was done in Remark B.3.2. All elements of \mathfrak{h} are semisimple, thus $\text{ad}(x)$ is diagonalizable for all $x \in \mathfrak{h}$ and since Eq. (B.3.2) holds, all endomorphisms $\text{ad}(x)$, for $x \in \mathfrak{h}$, are simultaneously diagonalizable. That means there is a basis $\{x_1, x_2, \dots\}$ of \mathfrak{g} such that

$$(\text{ad}(x))(x_i) = [x, x_i] = \alpha_{x_i}(x)x_i \quad (\text{B.3.3})$$

for $i = 1, 2, \dots$, where $\alpha_{x_i} \in \mathfrak{h}^*$ is the linear function $\mathfrak{h} \rightarrow \mathbb{k}$ that assigns an element $x \in \mathfrak{h}$ the eigenvalue $\alpha_{x_i}(x)$ of $\text{ad}(x)$ corresponding to the eigenvector x_i .

Definition B.3.3 (Root System) *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a semisimple Lie algebra over an algebraically closed field \mathbb{k} and $\{x_1, x_2, \dots\}$ a basis of \mathfrak{g} such that Eq. (B.3.3) holds. Then $\alpha_{x_1}, \alpha_{x_2}, \dots$ is said to be a weight system and α_{x_i} a weight of the adjoint representation of the Cartan subalgebra \mathfrak{h} . The elements $\alpha_{x_i} \neq 0$ are said to be the roots of \mathfrak{g} and the set of roots Δ is said to be a root system of \mathfrak{g} . We often omit the index x_i .*

Remark B.3.4 One can prove that the elements $x \in \mathfrak{g}$ which are weights but no roots of \mathfrak{g} are elements of \mathfrak{h} . Conversely it is clear that no root $x \in \mathfrak{h}$ can be a weight since $[h, x] = 0$ for all $h \in \mathfrak{h}$. Thus if we define for every root $\alpha \in \Delta$ the space

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\} \quad (\text{B.3.4})$$

there is a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad (\text{B.3.5})$$

of \mathfrak{g} . It is called the **root space decomposition** of \mathfrak{g} corresponding to the Cartan subalgebra \mathfrak{h} . The number $\dim(\mathfrak{g}_\alpha)$ is called the **multiplicity** of the root α .

The joint eigenspaces \mathfrak{g}_α have some properties that are easy to prove by calculation.

Proposition B.3.5 *The following statements hold for a root decomposition (B.3.5). Let $\alpha, \beta \in \mathfrak{h}^*$. Then*

- i.) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$,
- ii.) if $\alpha \neq 0$ and $x \in \mathfrak{g}_\alpha$ then $\text{ad}(x)$ is a nilpotent endomorphism,
- iii.) if $\alpha + \beta \neq 0$ then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\}$.

Let us come back to the Cartan decomposition (B.2.13) of a semisimple Lie algebra. We consider a root space decomposition of the eigenspace \mathfrak{p} corresponding to the -1 eigenvalue of a Cartan involution Θ . The Cartan subalgebra of \mathfrak{p} shall now be denoted by \mathfrak{a} . We are not interested in all roots Δ but only in the positive roots Δ^+ . Choose a basis $\{\alpha_1, \alpha_2, \dots\}$ of Δ . A root $\alpha \in \Delta$ is said to be *positive* if the first non-zero coefficient in the basis representation is positive. Then we define

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha. \quad (\text{B.3.6})$$

Definition B.3.6 (Iwasawa Decomposition) *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a real semisimple Lie algebra. The decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad (\text{B.3.7})$$

is said to be a Iwasawa decomposition of \mathfrak{g} . The Iwasawa decomposition at Lie group level is

$$G = KAN, \quad (\text{B.3.8})$$

where G, K, A and N denote the connected Lie groups corresponding to $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}$ and \mathfrak{n} respectively.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet und die Arbeit keiner anderen Prüfungsbehörde unter Erlangung eines akademischen Grades vorgelegt habe.

Würzburg, den 06.06.2016

Thomas Weber